S K Mondal's

Strength of Materials

Contents

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Note

"Asked Objective Questions" is the total collection of questions from:-

20 yrs IES (2010-1992) [Engineering Service Examination]

21 yrs. GATE (2011-1992)

and 14 yrs. IAS (Prelim.) [Civil Service Preliminary]

Every effort has been made to see that there are no errors (typographical or otherwise) in the material presented. However, it is still possible that there are a few errors (serious or otherwise). I would be thankful to the readers if they are brought to my attention at the following e-mail address: swapan_mondal_01@yahoo.co.in

S K Mondal

1. Stress and Strain

Theory at a Glance (for IES, GATE, PSU)

1.1 Stress (σ)

When a material is subjected to an external force, a resisting force is set up within the component. The internal resistance force per unit area acting on a material or intensity of the forces distributed over a given section is called the stress at a point.

- It uses original cross section area of the specimen and also known as engineering stress or conventional stress.

  Therefore, $\sigma = \dfrac{P}{A}$

- *P is* expressed in *Newton* (N) and *A,* original area, in square meters (m^2), the stress σ will be expresses in N/ m^2. This unit is called *Pascal (Pa).*
- As *Pascal* is a small quantity, in practice, multiples of this unit is used.

  1 kPa = 10^3 Pa = 10^3 N/ m^2 (kPa = Kilo Pascal)

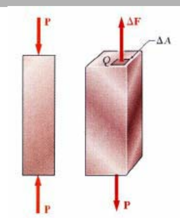
  1 MPa = 10^6 Pa = 10^6 N/ m^2 = 1 N/mm^2 (MPa = Mega Pascal)

  1 GPa = 10^9 Pa = 10^9 N/ m^2 (GPa = Giga Pascal)

Let us take an example: A rod 10 mm × 10 mm cross-section is carrying an axial tensile load 10 kN. In this rod the tensile stress developed is given by

$$(\sigma_t) = \frac{P}{A} = \frac{10kN}{(10mm \times 10mm)} = \frac{10 \times 10^3 N}{100mm^2} = 100\text{N/mm}^2 = 100\text{MPa}$$

- The resultant of the internal forces for an axially loaded member is normal to a section cut perpendicular to the member axis.

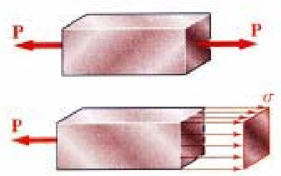


- The force intensity on the shown section is defined as the normal stress.

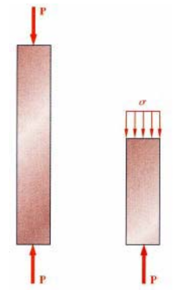
$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \quad \text{and} \quad \sigma_{avg} = \frac{P}{A}$$

- **Tensile stress (σ_t)**

If σ > 0 the stress is tensile. i.e. The fibres of the component tend to elongate due to the external force. A member subjected to an external force tensile P and tensile stress distribution due to the force is shown in the given figure.

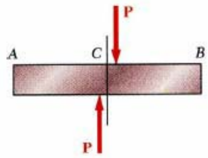


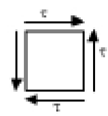
- **Compressive stress (σ_c)**



If σ < 0 the stress is compressive. i.e. The fibres of the component tend to shorten due to the external force. A member subjected to an external compressive force P and compressive stress distribution due to the force is shown in the given figure.

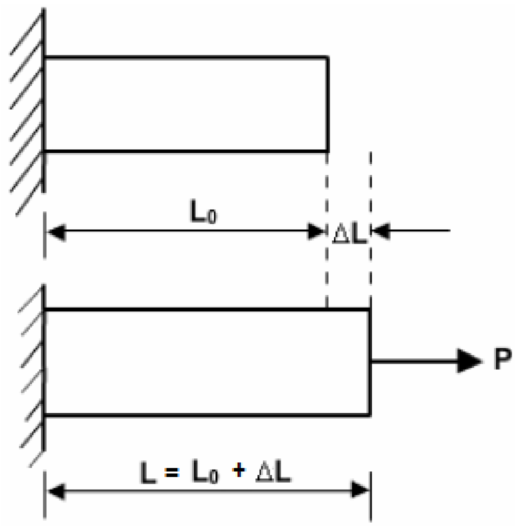
- **Shear stress (τ)**





When forces are transmitted from one part of a body to other, the stresses developed in a plane parallel to the applied force are the shear stress. **Shear stress acts parallel to plane of interest. Forces *P* is applied transversely to the member *AB as shown. The* corresponding internal forces act in the plane of section *C* and are called *shearing* forces. The corresponding average shear stress** $(\tau) = \frac{P}{Area}$

1.2 Strain (ε)



The displacement per unit length ***(dimensionless)*** is known as strain.

- **Tensile strain (ε_t)**

The elongation per unit length as shown in the figure is known as tensile strain.

$$\varepsilon_t = \Delta L / L_o$$

It is engineering strain or conventional strain.

Here we divide the elongation to original length not actual length ($L_o + \Delta L$)

Let us take an example: A rod 100 mm in original length. When we apply an axial tensile load 10 kN the final length of the rod after application of the load is 100.1 mm. So in this rod tensile strain is developed and is given by

$$(\varepsilon_t) = \frac{\Delta L}{L_o} = \frac{L - L_o}{L_o} = \frac{100.1mm - 100mm}{100mm} = \frac{0.1mm}{100mm} = 0.001 \text{ (Dimensionless) Tensile}$$

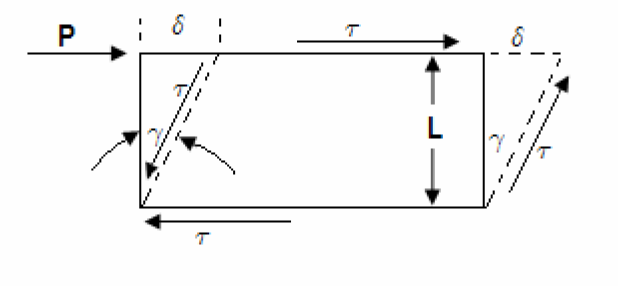
- **Compressive strain (ε_c)**

If the applied force is compressive then the reduction of length per unit length is known as compressive strain. It is negative. Then $\varepsilon_c = (-\Delta L)/ L_o$

Let us take an example: A rod 100 mm in original length. When we apply an axial compressive load 10 kN the final length of the rod after application of the load is 99 mm. So in this rod a compressive strain is developed and is given by

$$(\varepsilon_c) = \frac{\Delta L}{L_o} = \frac{L - L_o}{L_o} = \frac{99\,mm - 100\,mm}{100\,mm} = \frac{-1\,mm}{100\,mm} = -0.01 \text{ (Dimensionless) compressive}$$

- **Shear Strain (γ):** When a force P is applied tangentially to the element shown. Its edge displaced to dotted line. Where δ is the lateral displacement of the upper face



of the element relative to the lower face and L is the distance between these faces.

Then the shear strain is $(\gamma) = \dfrac{\delta}{L}$

Let us take an example: A block 100 mm × 100 mm base and 10 mm height. When we apply a tangential force 10 kN to the upper edge it is displaced 1 mm relative to lower face.

Then the direct shear stress in the element

$$(\tau) = \frac{10\,kN}{100\,mm \times 100\,mm} = \frac{10 \times 10^3\,N}{100\,mm \times 100\,mm} = 1 \text{ N/mm}^2 = 1 \text{ MPa}$$

And shear strain in the element (γ) = $= \dfrac{1\,mm}{10\,mm} = 0.1$ Dimensionless

1.3 True stress and True Strain

The true stress is defined as the ratio of the load to the cross section area at any instant.

$$(\sigma_T) = \frac{\text{load}}{\text{Instantaneous area}} = \sigma(1 + \varepsilon)$$

Where σ and ε is the engineering stress and engineering strain respectively.

- **True strain**

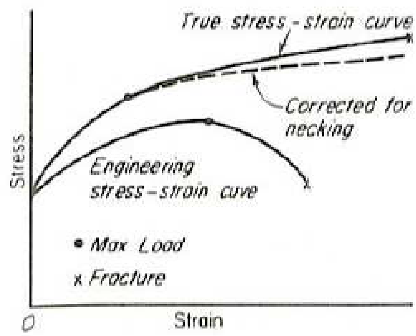
$$(\varepsilon_T) = \int_{L_o}^{L} \frac{dl}{l} = \ln\left(\frac{L}{L_o}\right) = \ln(1+\varepsilon) = \ln\left(\frac{A_o}{A}\right) = 2\ln\left(\frac{d_o}{d}\right)$$

or engineering strain (ε) = e^{ε_T} -1

The volume of the specimen is assumed to be constant during plastic deformation. [$\because A_o L_o = AL$] It is valid till the neck formation.

- **Comparison of engineering and the true stress-strain curves shown below**

- The **true stress-strain curve** is also known as the **flow curve.**
- True stress-strain curve gives a true indication of deformation characteristics because it is **based on the instantaneous dimension** of the specimen.
- In engineering stress-strain curve, stress drops down after necking since it is based on the original area.



- In true stress-strain curve, the stress however increases after necking since the cross-sectional area of the specimen decreases rapidly after necking.
- The flow curve of many metals in the region of uniform plastic deformation can be expressed by the **simple power law.**

$\sigma_T = K(\varepsilon_T)^n$ Where K is the strength coefficient

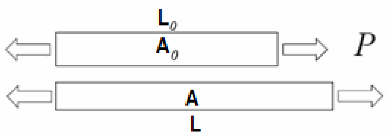
n is the strain hardening exponent

n = 0 perfectly plastic solid

n = 1 elastic solid

For most metals, 0.1< n < 0.5

- **Relation between the ultimate tensile strength and true stress at maximum load**



The ultimate tensile strength $(\sigma_u) = \dfrac{P_{max}}{A_o}$

The true stress at maximum load $(\sigma_u)_T = \dfrac{P_{max}}{A}$

And true strain at maximum load $(\varepsilon)_T = \ln\left(\dfrac{A_o}{A}\right)$ or $\dfrac{A_o}{A} = e^{\varepsilon_T}$

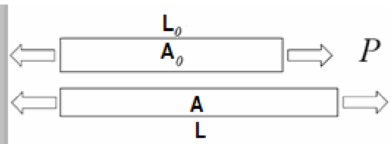
Eliminating P_{max} we get , $(\sigma_u)_T = \dfrac{P_{max}}{A} = \dfrac{P_{max}}{A_o} \times \dfrac{A_o}{A} = \sigma_u e^{\varepsilon_T}$

Where P_{max} = maximum force and A_o = Original cross section area

A = Instantaneous cross section area

Let us take two examples:

(I.) Only elongation no neck formation



In the tension test of a rod shown initially it was A_o = 50 mm² and L_o = 100 mm. After the application of load it's A = 40 mm² and L = 125 mm.

Determine the true strain using changes in both length and area.

Answer: First of all we have to check that does the member forms neck or not? For that check $A_oL_o = AL$ or not?

Here 50 × 100 = 40 × 125 so no neck formation is there. Therefore true strain

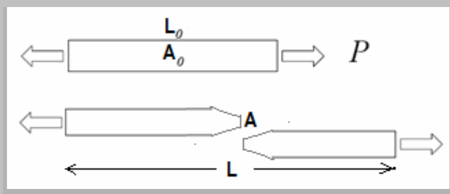
*(If **no neck formation** occurs both area and gauge length can be used for a strain calculation.)*

$$(\varepsilon_T) = \int_{L_o}^{L} \frac{dl}{l} = \ln\left(\frac{125}{100}\right) = 0.223$$

$$(\varepsilon_T) = \ln\left(\frac{A_o}{A}\right) = \ln\left(\frac{50}{40}\right) = 0.223$$

(II.) Elongation with neck formation

A ductile material is tested such and necking occurs then the final gauge length is L=140 mm and the final minimum cross sectional area is A = 35 mm². Though the rod shown initially it was A_o = 50 mm² and L_o = 100 mm. Determine the true strain using changes in both length and area.



Answer: First of all we have to check that does the member forms neck or not? For that check $A_o L_o = AL$ or not?

Here A_oL_o = 50 × 100 = 5000 mm³ and AL=35 × 140 = 4200 mm³. So neck formation is there. Note here $A_oL_o > AL$.

Therefore true strain

$$(\varepsilon_T) = \ln\left(\frac{A_o}{A}\right) = \ln\left(\frac{50}{35}\right) = 0.357$$

But not $(\varepsilon_T) = \int_{L_o}^{L} \frac{dl}{l} = \ln\left(\frac{140}{100}\right) = 0.336$ (it is wrong)

(***After necking,*** *gauge length gives error but area and diameter can be used for the calculation of true strain at fracture and before fracture also.*)

1.4 Hook's law

According to Hook's law the stress is directly proportional to strain i.e. normal stress (σ) α normal strain (ε) and shearing stress (τ) α shearing strain (γ).

$$\sigma = E\varepsilon \text{ and } \tau = G\gamma$$

The co-efficient E is called the ***modulus of elasticity*** *i.e. its resistance to elastic strain.* The co-efficient G is called the ***shear modulus of elasticity*** *or* ***modulus of rigidity.***

1.5 Volumetric strain (ε_v)

A relationship similar to that for length changes holds for three-dimensional (volume) change. For *volumetric strain,* (ε_v), the relationship is $(\varepsilon_v) = (V-V_0)/V_0$ or $(\varepsilon_v) = \Delta V/V_0 = \frac{P}{K}$

- Where V is the final volume, V_0 is the original volume, and ΔV is the volume change.
- Volumetric strain is a ratio of values with the same units, so it also is a dimensionless quantity.

- $\Delta V/V$= *volumetric strain* = $\varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$
- **Dilation:** The hydrostatic component of the total stress contributes to deformation by changing the area (or volume, in three dimensions) of an object. Area or volume change is called **dilation** and is positive or negative, as the volume increases or decreases, respectively. $e = \frac{p}{K}$ Where p is pressure.

1.6 Young's modulus or Modulus of elasticity (E) = $\frac{PL}{A\delta} = \frac{\sigma}{\in}$

1.7 Modulus of rigidity or Shear modulus of elasticity (G) = $\frac{\tau}{\gamma} = \frac{PL}{A\delta}$

1.8 Bulk Modulus or Volume modulus of elasticity (K) = $-\frac{\Delta p}{\frac{\Delta v}{v}} = \frac{\Delta p}{\frac{\Delta R}{R}}$

1.10 Relationship between the elastic constants E, G, K, μ

$$E = 2G(1+\mu) = 3K(1-2\mu) = \frac{9KG}{3K+G}$$ **[VIMP]**

Where K = Bulk Modulus, μ = Poisson's Ratio, E= Young's modulus, G= Modulus of rigidity

- For a linearly elastic, isotropic and homogeneous material, the number of elastic constants required to relate stress and strain is two. i.e. any two of the four must be known.
- If the material is non-isotropic (i.e. anisotropic), then the elastic modulii will vary with additional stresses appearing since there is a coupling between shear stresses and normal stresses for an anisotropic material.

Let us take an example: The modulus of elasticity and rigidity of a material are 200 GPa and 80 GPa, respectively. Find all other elastic modulus.

Answer: Using the relation $E = 2G(1+\mu) = 3K(1-2\mu) = \frac{9KG}{3K+G}$ we may find all other elastic modulus easily

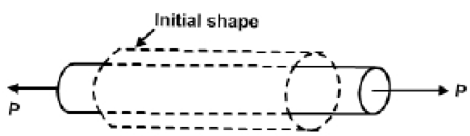
Poisson's Ratio (μ): $1+\mu = \frac{E}{2G}$ $\Rightarrow \mu = \frac{E}{2G} - 1 = \frac{200}{2\times 80} - 1 = 0.25$

Bulk Modulus (K) : $3K = \frac{E}{1-2\mu}$ $\Rightarrow K = \frac{E}{3(1-2\mu)} = \frac{200}{3(1-2\times 0.25)} = 133.33\,\text{GPa}$

1.11 Poisson's Ratio (μ)

$$= \frac{\text{Transverse strain or lateral strain}}{\text{Longitudinal strain}} = \frac{-\in_y}{\in_x}$$

(Under unidirectional stress in x-direction)



- The theory of isotropic elasticity allows Poisson's ratios in the range from -1 to 1/2.
- Poisson's ratio in various materials

Material	Poisson's ratio	Material	Poisson's ratio
Steel	0.25 – 0.33	Rubber	0.48 – 0.5
C.I	0.23 – 0.27	Cork	Nearly zero
Concrete	0.2	Novel foam	negative

- We use cork in a bottle as the cork easily inserted and removed, yet it also withstand the pressure from within the bottle. Cork with a Poisson's ratio of nearly zero, is ideal in this application.

1.12 For bi-axial stretching of sheet

$$\in_1 = \ln\left(\frac{L_{f1}}{L_{o1}}\right) \qquad L_o - \textit{Original} \text{ length}$$

$$\in_2 = \ln\left(\frac{L_{f2}}{L_{o2}}\right) \qquad L_f \text{-Final length}$$

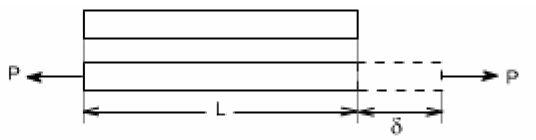
$$\text{Final thickness } (t_f) = \frac{\textit{Initial} \text{ thickness}(t_o)}{e^{\in_1} \times e^{\in_2}}$$

1.13 Elongation

- **A prismatic bar loaded in tension by an axial force P**

For a prismatic bar loaded in tension by an axial force P. The elongation of the bar can be determined as

$$\delta = \frac{PL}{AE}$$



Let us take an example: A Mild Steel wire 5 mm in diameter and 1 m long. If the wire is subjected to an axial tensile load 10 kN find its extension of the rod. (E = 200 GPa)

Answer: We know that $(\delta) = \frac{PL}{AE}$

Here given, $\text{Force}(P) = 10\,kN = 10 \times 1000\,\text{N}$

$\text{Length}(L) = 1\text{ m}$

$$\text{Area}(A) = \frac{\pi d^2}{4} = \frac{\pi \times (0.005)^2}{4}\text{m}^2 = 1.963 \times 10^{-5}\text{m}^2$$

Modulus of Elasticity $(E) = 200\ GPa = 200 \times 10^9\ \text{N/m}^2$

$$\text{Therefore Elongation}(\delta) = \frac{PL}{AE} = \frac{(10 \times 1000) \times 1}{(1.963 \times 10^{-5}) \times (200 \times 10^9)}\text{m}$$

$$= 2.55 \times 10^{-3}\text{ m} = 2.55\text{ mm}$$

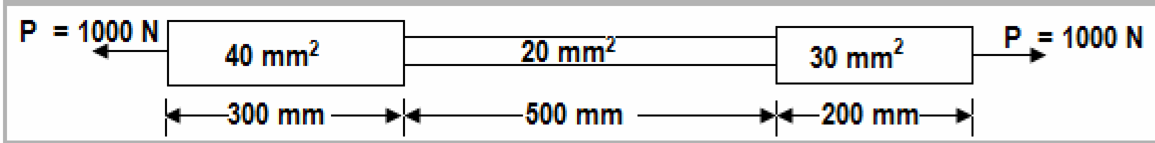
- ***Elongation of composite body***

Elongation of a bar of varying cross section $A_1, A_2, \text{---------}, A_n$ of lengths $l_1, l_2, \text{-------} l_n$ respectively.

$$\delta = \frac{P}{E}\left[\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3} - - - - - - - - + \frac{l_n}{A_n}\right]$$

Let us take an example: A composite rod is 1000 mm long, its two ends are 40 mm² and 30 mm² in area and length are 300 mm and 200 mm respectively. The middle portion of the rod is 20 mm² in area and 500 mm long. If the rod is subjected to an axial tensile load of 1000 N, find its total elongation. (E = 200 GPa).

Answer: Consider the following figure



Given, Load (P) =1000 N

Area; $(A_1) = 40\ mm^2$, $A_2 = 20\ mm^2$, $A_3 = 30\ mm^2$

Length; $(l_1) = 300$ mm, $l_2 = 500$ mm, $l_3 = 200$ mm

$E = 200\ GPa = 200 \times 10^9\ N/m^2 = 200 \times 10^3\ N/mm^2$

Therefore Total extension of the rod

$$\delta = \frac{P}{E}\left[\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right]$$

$$= \frac{1000N}{200 \times 10^3\ N/mm^2} \times \left[\frac{300mm}{40mm^2} + \frac{500mm}{20mm^2} + \frac{200mm}{30mm^2}\right]$$

$$= 0.196\,mm$$

- ***Elongation of a tapered body***

Elongation of a tapering rod of length 'L' due to load 'P' at the end

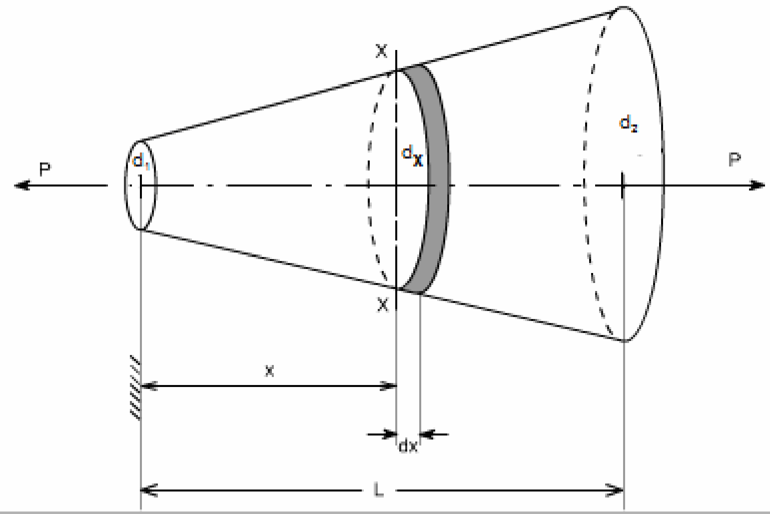
$$\boxed{\delta = \frac{4PL}{\pi E d_1 d_2}}$$ (d_1 and d_2 are the diameters of smaller & larger ends)

You may remember this in this way, $\delta = \dfrac{PL}{E\left(\dfrac{\pi}{4} d_1 d_2\right)}$ *i.e.* $\dfrac{PL}{EA_{eq}}$

Let us take an example: A round bar, of length L, tapers uniformly from small diameter d_1 at one end to bigger diameter d_2 at the other end. Show that the extension produced by a tensile axial load P is $(\delta) = \dfrac{4PL}{\pi d_1 d_2 E}$.

If $d_2 = 2d_1$, compare this extension with that of a uniform cylindrical bar having a diameter equal to the mean diameter of the tapered bar.

Answer: Consider the figure below d_1 be the radius at the smaller end. Then at a X cross section XX located at a distance × from the smaller end, the value of diameter 'd_x' is equal to



$$\frac{d_x}{2} = \frac{d_1}{2} + \frac{x}{L}\left(\frac{d_2}{2} - \frac{d_1}{2}\right)$$

$$or\ d_x = d_1 + \frac{x}{L}(d_2 - d_1)$$

$$= d_1(1+kx) \qquad Where\ k = \frac{d_2 - d_1}{L} \times \frac{1}{d_1}$$

We now taking a small strip of diameter 'd_x'and length 'd_x'at section XX.
Elongation of this section 'd_x' length

$$d(\delta) = \frac{PL}{AE} = \frac{P.dx}{\left(\frac{\pi d_x^2}{4}\right)\times E} = \frac{4P.dx}{\pi.\{d_1(1+kx)\}^2 E}$$

Therefore total elongation of the taper bar

$$\delta = \int d(\delta) = \int_{x=0}^{x=L} \frac{4P\ dx}{\pi E d_1^2 (1+kx)^2}$$

$$= \frac{4PL}{\pi E\, d_1 d_2}$$

Comparison: Case-I: Where $d_2 = 2d_1$

$$\text{Elongation}(\delta_I) = \frac{4PL}{\pi E d_1 \times 2d_1} = \frac{2PL}{\pi E d_1^2}$$

Case –II: Where we use Mean diameter

$$d_m = \frac{d_1 + d_2}{2} = \frac{d_1 + 2d_1}{2} = \frac{3}{2}d_1$$

$$\text{Elongation of such bar } (\delta_{II}) = \frac{PL}{AE} = \frac{P.L}{\frac{\pi}{4}\left(\frac{3}{2}d_1\right)^2 .E}$$

$$= \frac{16PL}{9\pi E d_1^2}$$

$$\frac{\text{Extension of taper bar}}{\text{Extension of uniform bar}} = \frac{2}{\frac{16}{9}} = \frac{9}{8}$$

- ***Elongation of a body due to its self weight***

(i) Elongation of a uniform rod of length 'L' due to its own weight 'W'

$$\delta=\frac{WL}{2AE}$$

The deformation of a bar under its own weight as compared to that when subjected to a direct axial load equal to its own weight ***will be half***.

(ii) Total extension produced in rod of length 'L' due to its own weight 'ω' per with

length. $\delta=\frac{\omega L^2}{2EA}$

(iii) Elongation of a conical bar due to its self weight

$$\delta=\frac{\rho g L^2}{6E}=\frac{WL}{2A_{\max}E}$$

1.14 Structural members or machines must be designed such that the working stresses are less than the ultimate strength of the material.

$$\left.\begin{aligned} \text{Working stress}(\sigma_w) &= \frac{\sigma_y}{n} \quad \text{n=1.5 to 2} \\ &= \frac{\sigma_{ult}}{n_1} \quad n_1 = 2 \text{ to } 3 \end{aligned}\right\} \text{factor of safety}$$

$$= \frac{\sigma_p}{n} \qquad \sigma_p = \text{Proof stress}$$

1.15 Factor of Safety: (n) = $\frac{\sigma_y \ or \ \sigma_p \ or \ \sigma_{ult}}{\sigma_w}$

1.16 Thermal or Temperature stress and strain

- When a material undergoes a change in temperature, it either elongates or contracts depending upon whether temperature is increased or decreased of the material.
- If the elongation or contraction is *not restricted*, i. e. *free* then the material does not experience *any stress despite the fact that it undergoes a strain.*
- The strain due to temperature change is called *thermal strain* and is expressed as,

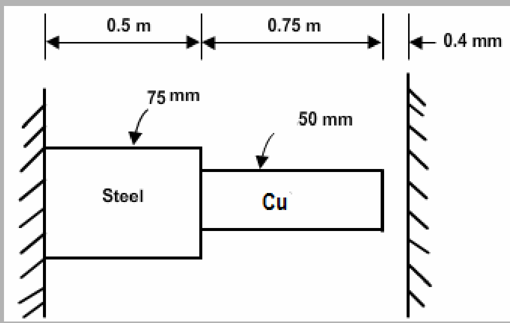
$$\varepsilon = \alpha(\Delta T)$$

- Where α is co-efficient of thermal expansion, a material property, and ΔT is the change in temperature.
- The free expansion or contraction of materials, when restrained induces stress in the material and it is referred to as *thermal stress*.

$$\sigma_t = \alpha E(\Delta T)$$ Where, E = Modulus of elasticity

- Thermal stress produces the same effect in the material similar to that of mechanical stress. A compressive stress will produce in the material with increase in temperature and the stress developed is tensile stress with decrease in temperature.

Let us take an example: A rod consists of two parts that are made of steel and copper as shown in figure below. The elastic modulus and coefficient of thermal expansion for steel are 200 GPa and 11.7×10^{-6} per °C respectively and for copper 70 GPa and 21.6×10^{-6} per °C respectively. If the temperature of the rod is raised by 50°C, determine the forces and stresses acting on the rod.



Answer: If we allow this rod to freely expand then free expansion

$$\begin{aligned}\delta_T &= \alpha(\Delta T)L \\ &= \left(11.7\times10^{-6}\right)\times50\times500+\left(21.6\times10^{-6}\right)\times50\times750 \\ &= 1.1025 \text{ mm (Compressive)}\end{aligned}$$

But according to diagram only free expansion is 0.4 mm.

Therefore restrained deflection of rod =1.1025 mm – 0.4 mm = 0.7025 mm

Let us assume the force required to make their elongation vanish be P which is the reaction force at the ends.

$$\delta = \left(\frac{PL}{AE}\right)_{Steel} + \left(\frac{PL}{AE}\right)_{Cu}$$

$$or \quad 0.7025 = \frac{P\times500}{\left\{\frac{\pi}{4}\times(0.075)^2\right\}\times\left(200\times10^9\right)} + \frac{P\times750}{\left\{\frac{\pi}{4}\times(0.050)^2\right\}\times\left(70\times10^9\right)}$$

$$or\ P = 116.6\ kN$$

Therefore, compressive stress on steel rod

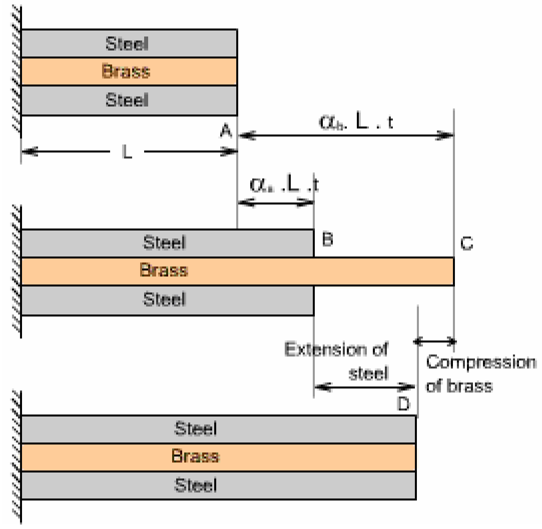
$$\sigma_{Steel} = \frac{P}{A_{Steel}} = \frac{116.6\times10^3}{\frac{\pi}{4}\times(0.075)^2}\text{N/m}^2 = 26.39 \text{ MPa}$$

And compressive stress on copper rod

$$\sigma_{Cu} = \frac{P}{A_{Cu}} = \frac{116.6\times10^3}{\frac{\pi}{4}\times(0.050)^2}\text{N/m}^2 = 59.38 \text{ MPa}$$

1.17 Thermal stress on Brass and Mild steel combination

A brass rod placed within a steel tube of exactly same length. The assembly is making in such a way that elongation of the combination will be same. To calculate the stress induced in the brass rod, steel tube when the combination is raised by t^oC then the following analogy have to do.



(a) Original bar before heating.

(b) Expanded position if the members are allowed to expand freely and independently after heating.

(c) Expanded position of the compound bar i.e. final position after heating.

- Compatibility Equation:

$$\delta = \delta_{st} + \delta_{sf} = \delta_{Bt} - \delta_{Bf}$$

- Equilibrium Equation:

$$\sigma_s A_s = \sigma_B A_B$$

Assumption:

1. $L = L_s = L_B$
2. $\alpha_b > \alpha_s$
3. $Steel - Tension$
   $Brass - Compression$

Where, δ = Expansion of the compound bar = AD in the above figure.

δ_{st} = Free expansion of the steel tube due to temperature rise t^oC = $\alpha_s L t$

= AB in the above figure.

δ_{sf} = Expansion of the steel tube due to internal force developed by the unequal expansion.

= BD in the above figure.

δ_{Bt} = Free expansion of the brass rod due to temperature rise t^oC = $\alpha_b L t$

= AC in the above figure.

δ_{Bf} = Compression of the brass rod due to internal force developed by the unequal expansion.

= BD in the above figure.

And in the equilibrium equation

Tensile force in the steel tube = Compressive force in the brass rod

Where, σ_s = Tensile stress developed in the steel tube.

σ_B = Compressive stress developed in the brass rod.

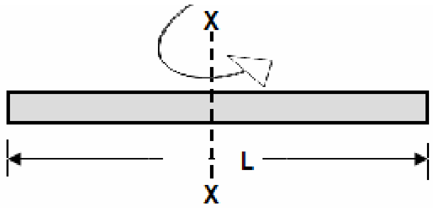
A_s = Cross section area of the steel tube.

A_B = Cross section area of the brass rod.

Let us take an example: See the Conventional Question Answer section of this chapter and the question is "***Conventional Question IES-2008***" and it's answer.

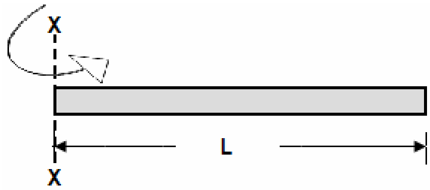
1.18 Maximum stress and elongation due to rotation

(i) $\sigma_{\max} = \dfrac{\rho\omega^2 L^2}{8}$ and $(\delta L) = \dfrac{\rho\omega^2 L^3}{12E}$



(ii) $\sigma_{\max} = \dfrac{\rho\omega^2 L^2}{2}$ and $(\delta L) = \dfrac{\rho\omega^2 L^3}{3E}$

For remember: You will get (ii) by multiplying by 4 of (i)

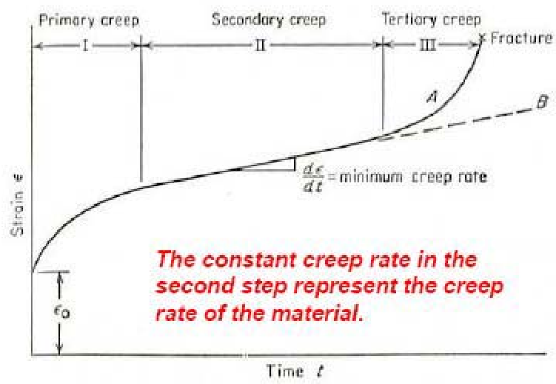


1.18 Creep

When a member is subjected to a constant load over a long period of time it undergoes a slow permanent deformation and this is termed as "creep". This is dependent on temperature. Usually at elevated temperatures creep is high.

- The materials have its own different melting point; each will creep when the homologous temperature > 0.5. Homologous temp = $\dfrac{\text{Testing temperature}}{\text{Melting temperature}} > 0.5$

A typical creep curve shows three distinct stages with different creep rates. After an initial rapid elongation ε_o, the creep rate decrease with time until reaching the steady state.



1) ***Primary creep*** is a period of transient creep. The creep resistance of the material increases due to material deformation.

2) ***Secondary creep*** provides a nearly constant creep rate. The average value of the creep rate during this period is called the minimum creep rate. A stage of balance between competing.

Strain hardening and ***recovery*** (softening) of the material.

3) ***Tertiary creep*** shows a rapid increase in the creep rate due to effectively reduced cross-sectional area of the specimen leading to *creep rupture* or failure. In this stage *intergranular* cracking and/or formation of voids and cavities occur.

Creep rate $= c_1 \sigma^{c_2}$

Creep strain at any time = zero time strain intercept + creep rate ×Time

$$= \epsilon_0 + c_1 \sigma^{c_2} \times t$$

Where, c_1, c_2 *are* constants $\sigma = stress$

1.19 If a load P is applied suddenly to a bar then the stress & strain induced will be ***double*** than those obtained by an equal load applied gradually.

1.20 Stress produced by a load P in falling from height 'h'

$$\sigma_d = \sigma\left[1+\sqrt{1+\frac{2h}{\in L}}\right]\Bigg\}\ \sigma,$$

$\in$ being stress & strain produced by static load P & L=length of bar.

$$= \frac{A}{P}\left[1+\sqrt{1+\frac{2AEh}{PL}}\right]$$

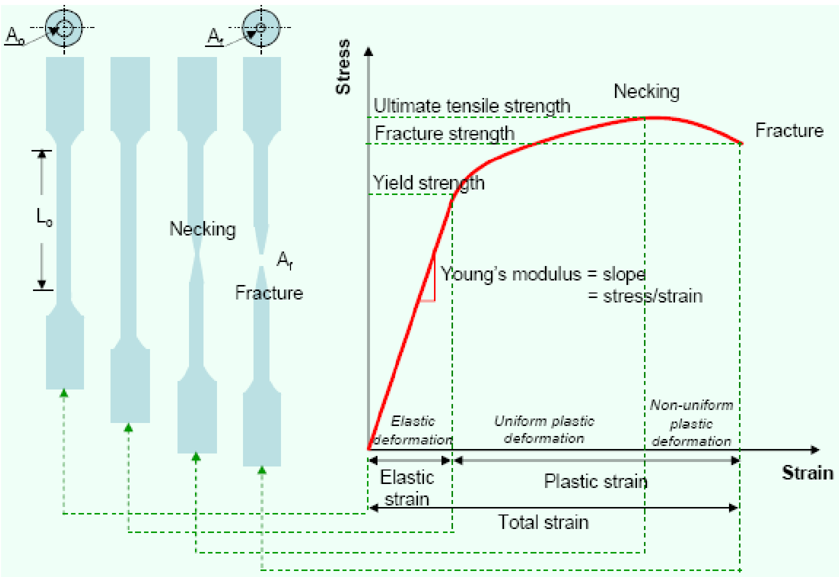
1.21 Loads shared by the materials of a compound bar made of bars x & y due to load W,

$$P_x = W.\frac{A_xE_x}{A_xE_x + A_yE_y}$$

$$P_y = W.\frac{A_yE_y}{A_xE_x + A_yE_y}$$

1.22 Elongation of a compound bar, $\delta = \dfrac{PL}{A_xE_x + A_yE_y}$

1.23 Tension Test

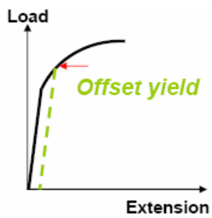


i) **True elastic limit:** based on micro-strain measurement at strains on order of 2 × 10^{-6}. Very low value and is related to the motion of a few hundred dislocations.

ii) **Proportional limit:** the highest stress at which stress is directly proportional to strain.

iii) **Elastic limit:** is the greatest stress the material can withstand without any measurable permanent strain after unloading. Elastic limit > proportional limit.

iv) **Yield strength** is the stress required to produce a small specific amount of deformation. The offset yield strength can be determined by the stress corresponding to the intersection of the stress-strain curve and a line parallel to the elastic line offset by a strain of 0.2 or 0.1%. (ε = 0.002 or 0.001).



- The offset yield stress is referred to proof stress either at 0.1 or 0.5% strain used for design and specification purposes to avoid the practical difficulties of measuring the elastic limit or proportional limit.

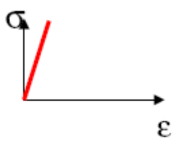
v) Tensile strength or ultimate tensile strength (UTS) σ_u is the maximum load P_{max} divided by the original cross-sectional area A_o of the specimen.

vi) % Elongation, $= \dfrac{L_f - L_o}{L_o}$**,** is chiefly influenced by uniform elongation, which is dependent on the strain-hardening capacity of the material.

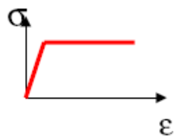
vii) Reduction of Area: $q = \dfrac{A_o - A_f}{A_o}$

- Reduction of area is more a measure of the deformation required to produce failure and its chief contribution results from the necking process.
- Because of the complicated state of stress state in the neck, values of reduction of area are dependent on specimen geometry, and deformation behaviour, and they should not be taken as true material properties.
- RA is the most structure-sensitive ductility parameter and is useful in detecting quality changes in the materials.

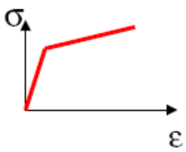
viii) Stress-strain response



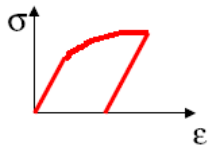
Linear elastic



Linear elastic-perfectly plastic



Linear elastic-hardening plastic



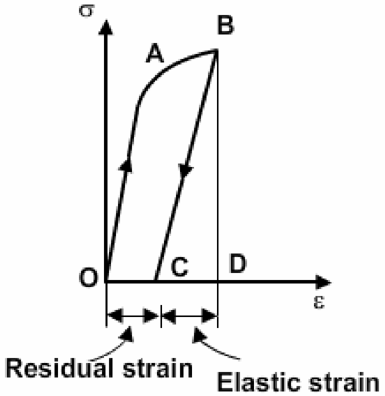
Linear elastic-hardening plasticity with unloading

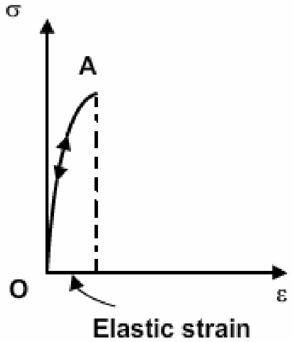
1.24 Elastic strain and Plastic strain

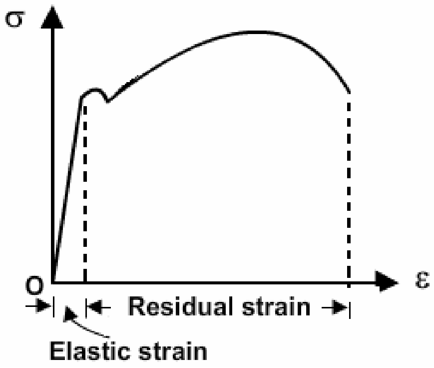
The strain present in the material after unloading is called the **residual strain or plastic strain** and the strain disappears during unloading is termed as **recoverable or elastic strain.**
Equation of the straight line CB is given by

$$\sigma = \in_{total} \times E - \in_{Plastic} \times E = \in_{Elastic} \times E$$

Carefully observe the following figures and understand which one is Elastic strain and which one is Plastic strain







Let us take an example: A 10 mm diameter tensile specimen has a 50 mm gauge length. The load corresponding to the 0.2% offset is 55 kN and the maximum load is 70 kN. Fracture occurs at 60 kN. The diameter after fracture is 8 mm and the gauge length at fracture is 65 mm. Calculate the following properties of the material from the tension test.

(i) % Elongation
(ii) Reduction of Area (RA) %
(iii) Tensile strength or ultimate tensile strength (UTS)
(iv) Yield strength
(v) Fracture strength
(vi) If E = 200 GPa, the elastic recoverable strain at maximum load
(vii) If the elongation at maximum load (the uniform elongation) is 20%, what is the plastic strain at maximum load?

Answer: Given, Original area $\left(A_0\right) = \frac{\pi}{4}\times\left(0.010\right)^2 \text{ m}^2 = 7.854\times10^{-5}\text{m}^2$

Area at fracture $\left(A_f\right) = \frac{\pi}{4}\times\left(0.008\right)^2 \text{ m}^2 = 5.027\times10^{-5}\text{m}^2$

Original gauge length (L_0) = 50 mm

Gauge length at fracture (L) = 65 mm

Therefore

(i) % Elongation $= \frac{L-L_0}{L_0}\times100\% = \frac{65-50}{50}\times100 = 30\%$

(ii) Reduction of area (RA) = $q = \frac{A_0-A_f}{A_0}\times100\% = \frac{7.854-5.027}{7.854}\times100\% = 36\%$

(iii) Tensile strength or Ultimate tensile strength (UTS), $\sigma_u = \frac{P_{max}}{A_o} = \frac{70\times10^3}{7.854\times10^{-5}}\text{N/m}^2 = 891\,\text{MPa}$

(iv) Yield strength $\left(\sigma_y\right) = \frac{P_y}{A_o} = \frac{55\times10^3}{7.854\times10^{-5}}\text{N/m}^2 = 700\text{ MPa}$

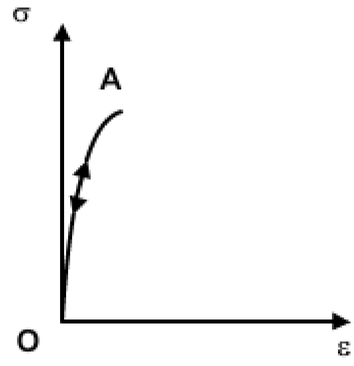
(v) Fracture strength $\left(\sigma_F\right) = \frac{P_{Fracture}}{A_o} = \frac{60\times10^3}{7.854\times10^{-5}}\text{N/m}^2 = 764\,\text{MPa}$

(vi) Elastic recoverable strain at maximum load $\left(\varepsilon_E\right) = \frac{P_{\max}/A_o}{E} = \frac{891\times10^6}{200\times10^9} = 0.0045$

(vii) Plastic strain $(\varepsilon_P) = \varepsilon_{total} - \varepsilon_E = 0.2000 - 0.0045 = 0.1955$

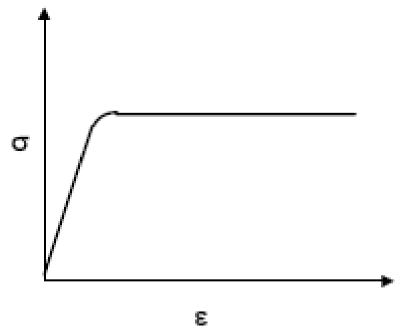
1.25 Elasticity

This is the property of a material to regain its original shape after deformation when the external forces are removed. When the material is in elastic region the strain disappears completely after removal of the load, The stress-strain relationship in elastic region need not be linear and can be non-linear (example rubber). The maximum stress value below which the strain is fully recoverable is called the *elastic limit*. It is represented by point A in figure. All materials are elastic to some extent but the degree varies, for example, both mild steel and rubber are elastic materials but steel is more elastic than rubber.



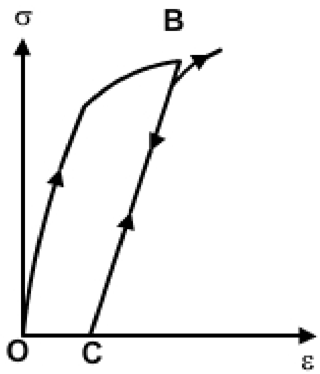
1.26 Plasticity

When the stress in the material exceeds the elastic limit, the material enters into plastic phase where the strain can no longer be completely removed. Under plastic conditions materials ideally deform without any increase in stress. A typical stress strain diagram for an elastic-perfectly plastic material is shown in the figure. Mises-Henky criterion gives a good starting point for plasticity analysis.



1.27 Strain hardening

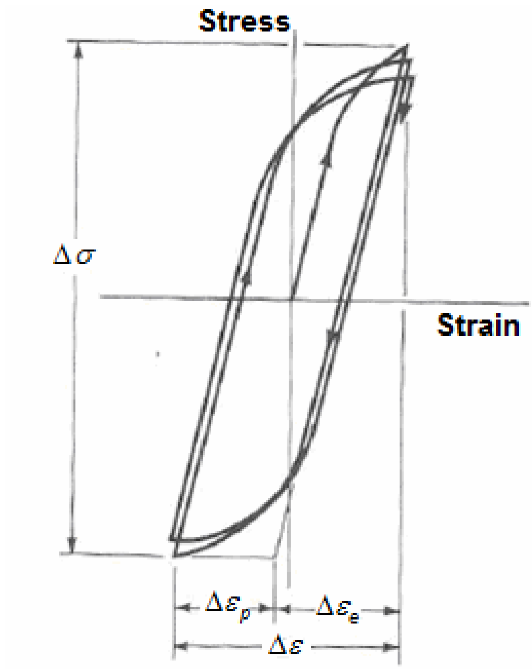
If the material is reloaded from point C, it will follow the previous unloading path and line CB becomes its new elastic region with elastic limit defined by point B. Though the new elastic region CB resembles that of the initial elastic region OA, the internal structure of the material in the new state has changed. The change in the microstructure of the material is clear from the fact that the ductility of the material has come down due to strain hardening. When the material is reloaded, it follows the same path as that of a virgin material and fails on reaching the ultimate strength which remains unaltered due to the intermediate loading and unloading process.



1.28 Stress reversal and stress-strain hysteresis loop

We know that fatigue failure begins at a local discontinuity and when the stress at the discontinuity exceeds elastic limit there is plastic strain. The cyclic plastic strain results crack propagation and fracture.

When we plot the experimental data with reversed loading and the true stress strain hysteresis loops is found as shown below.



True stress-strain plot with a number of stress reversals

Due to cyclic strain the elastic limit increases for annealed steel and decreases for cold drawn steel.

Here the stress range is $\Delta\sigma$. $\Delta\varepsilon_p$ and $\Delta\varepsilon_e$ are the plastic and elastic strain ranges, the total strain range being $\Delta\varepsilon$. Considering that the total strain amplitude can be given as

$$\Delta\varepsilon = \Delta\varepsilon_p + \Delta\varepsilon_e$$

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Stress in a bar due to self-weight

GATE-1. Two identical circular rods of same diameter and same length are subjected to same magnitude of axial tensile force. One of the rods is made out of mild steel having the modulus of elasticity of 206 GPa. The other rod is made out of cast iron having the modulus of elasticity of 100 GPa. Assume both the materials to be homogeneous and isotropic and the axial force causes the same amount of uniform stress in both the rods. The stresses developed are within the proportional limit of the respective materials. Which of the following observations is correct? [GATE-2003]

(a) Both rods elongate by the same amount
(b) Mild steel rod elongates more than the cast iron rod
(c) Cast iron rod elongates more than the mild steel rod
(d) As the stresses are equal strains are also equal in both the rods

GATE-1. Ans. (c) $\delta L = \dfrac{PL}{AE}$ or $\delta L \propto \dfrac{1}{E}$ [AsP, L and A is same]

$$\frac{(\delta L)_{\text{mild steel}}}{(\delta L)_{C.I}} = \frac{E_{CI}}{E_{MS}} = \frac{100}{206} \qquad \therefore (\delta L)_{CI} > (\delta L)_{MS}$$

GATE-2. A steel bar of 40 mm × 40 mm square cross-section is subjected to an axial compressive load of 200 kN. If the length of the bar is 2 m and E = 200 GPa, the elongation of the bar will be: [GATE-2006]

(a) 1.25 mm (b) 2.70 mm (c) 4.05 mm (d) 5.40 mm

GATE-2. Ans. (a) $\delta L = \dfrac{PL}{AE} = \dfrac{(200\times 1000)\times 2}{(0.04\times 0.04)\times 200\times 10^{9}}\,m = 1.25\,mm$

True stress and true strain

GATE-3. The ultimate tensile strength of a material is 400 MPa and the elongation up to maximum load is 35%. If the material obeys power law of hardening, then the true stress-true strain relation (stress in MPa) in the plastic deformation range is: [GATE-2006]

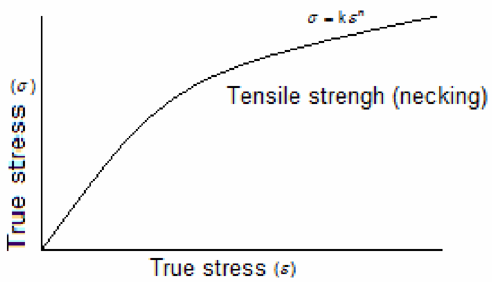
(a) $\sigma = 540\varepsilon^{0.30}$ (b) $\sigma = 775\varepsilon^{0.30}$ (c) $\sigma = 540\varepsilon^{0.35}$ (d) $\sigma = 775\varepsilon^{0.35}$

GATE-3. Ans. (c)

A true stress – true strain curve in tension $\sigma = k\varepsilon^{n}$

k = Strength co-efficient = 400 × (1.35) = 540 MPa

n = Strain – hardening exponent = 0.35



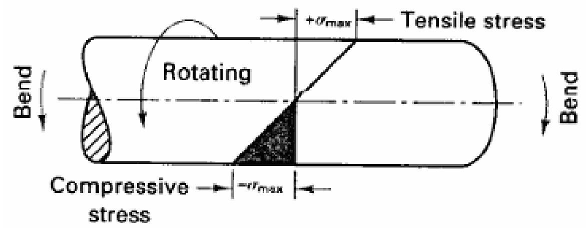
Elasticity and Plasticity

GATE-4. An axial residual compressive stress due to a manufacturing process is present on the outer surface of a rotating shaft subjected to bending. Under a given

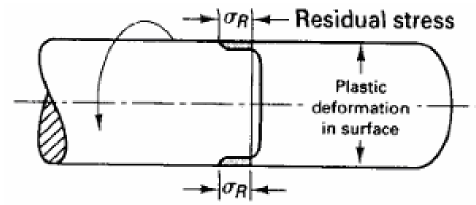
bending load, the fatigue life of the shaft in the presence of the residual compressive stress is: [GATE-2008]

(a) Decreased
(b) Increased or decreased, depending on the external bending load
(c) Neither decreased nor increased
(d) Increased

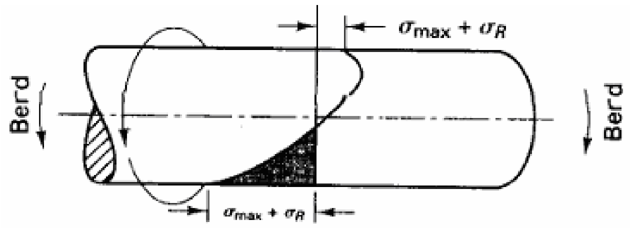
GATE-4. Ans. (d)



A cantilever-loaded rotating beam, showing the normal distribution of surface stresses. (i.e., tension at the top and compression at the bottom)



The residual compressive stresses induced.



Net stress pattern obtained when loading a surface treated beam. The reduced magnitude of the tensile stresses contributes to increased fatigue life.

GATE-5. A static load is mounted at the centre of a shaft rotating at uniform angular velocity. This shaft will be designed for [GATE-2002]

(a) The maximum compressive stress (static)
(b) The maximum tensile stress (static)
(c) The maximum bending moment (static)
(d) Fatigue loading

GATE-5. Ans. (d)

GATE-6. Fatigue strength of a rod subjected to cyclic axial force is less than that of a rotating beam of the same dimensions subjected to steady lateral force because

(a) Axial stiffness is less than bending stiffness **[GATE-1992]**
(b) Of absence of centrifugal effects in the rod
(c) The number of discontinuities vulnerable to fatigue are more in the rod
(d) At a particular time the rod has only one type of stress whereas the beam has both the tensile and compressive stresses.

GATE-6. Ans. (d)

Relation between the Elastic Modulii

GATE-7. A rod of length *L* and diameter *D* is subjected to a tensile load *P*. Which of the following is sufficient to calculate the resulting change in diameter?

(a) Young's modulus
(b) Shear modulus **[GATE-2008]**
(c) Poisson's ratio
(d) Both Young's modulus and shear modulus

GATE-7. Ans. (d) For longitudinal strain we need Young's modulus and for calculating transverse strain we need Poisson's ratio. We may calculate Poisson's ratio from $E = 2G(1+\mu)$ for that we need Shear modulus.

GATE-8. In terms of Poisson's ratio (μ) the ratio of Young's Modulus (E) to Shear Modulus (G) of elastic materials is [GATE-2004]

$(a)\, 2(1+\mu)$ $(b)\, 2(1-\mu)$ $(c)\, \frac{1}{2}(1+\mu)$ $(d)\, \frac{1}{2}(1-\mu)$

GATE-8. Ans. (a)

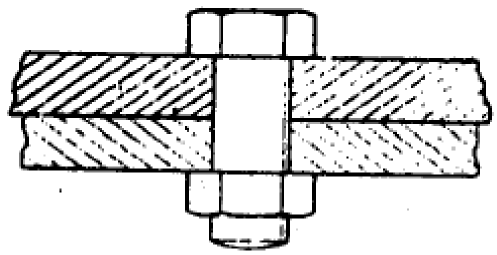
GATE-9. The relationship between Young's modulus (E), Bulk modulus (K) and Poisson's ratio (μ) is given by: [GATE-2002]

(a) $E = 3K(1-2\mu)$ (b) $K = 3E(1-2\mu)$
(c) $E = 3K(1-\mu)$ (d) $K = 3E(1-\mu)$

GATE-9. Ans. (a) Remember $E = 2G(1+\mu) = 3K(1-2\mu) = \dfrac{9KG}{3K+G}$

Stresses in compound strut

GATE-10. In a bolted joint two members are connected with an axial tightening force of 2200 N. If the bolt used has metric threads of 4 mm pitch, then torque required for achieving the tightening force is

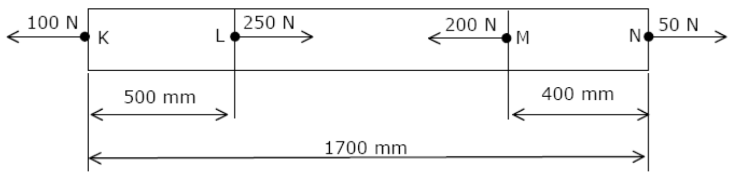


(a) 0.7Nm (b) 1.0 Nm
(c) 1.4Nm (d) 2.8Nm

[GATE-2004]

GATE-10. Ans. (c) $T = F \times r = 2200 \times \dfrac{0.004}{2\pi} \text{Nm} = 1.4\text{Nm}$

GATE-11. The figure below shows a steel rod of 25 mm² cross sectional area. It is loaded at four points, K, L, M and N. [GATE-2004, IES 1995, 1997, 1998]

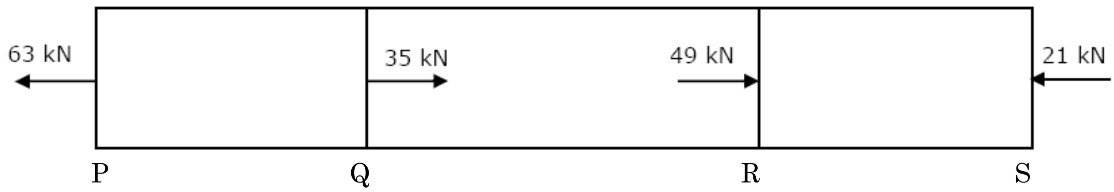


Assume E_{steel} = 200 GPa. The total change in length of the rod due to loading is:
(a) 1 μm (b) -10 μm (c) 16 μm (d) -20 μm

GATE-11. Ans. (b) First draw FBD of all parts separately then

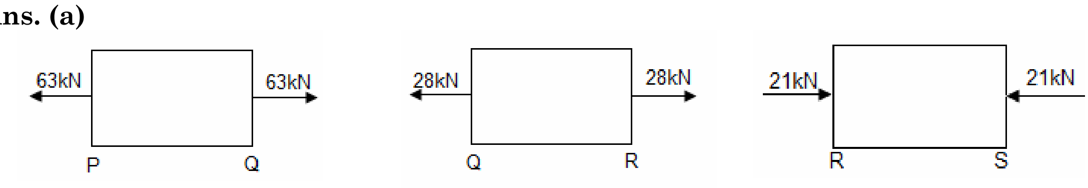
Total change in length = $\sum \dfrac{PL}{AE}$

GATE-12. A bar having a cross-sectional area of 700mm² is subjected to axial loads at the positions indicated. The value of stress in the segment QR is: [GATE-2006]



(a) 40 MPa (b) 50 MPa (c) 70 MPa (d) 120 MPa

GATE-12. Ans. (a)



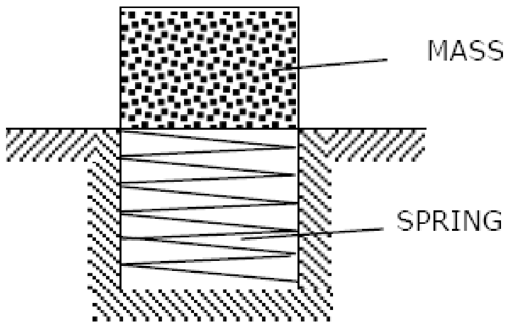
F.B.D

$$\sigma_{QR} = \frac{P}{A} = \frac{28000}{700}\text{MPa} = 40\text{MPa}$$

GATE-13. An ejector mechanism consists of a helical compression spring having a spring constant of K = 981 × 10³ N/m. It is pre-compressed by 100 mm from its free state. If it is used to eject a mass of 100 kg held on it, the mass will move up through a distance of

(a) 100mm (b) 500mm

(c) 981 mm (d) 1000mm



[GATE-2004]

GATE-13. Ans. (a) No calculation needed it is pre-compressed by 100 mm from its free state. So it can't move more than 100 mm. choice (b), (c) and (d) out.

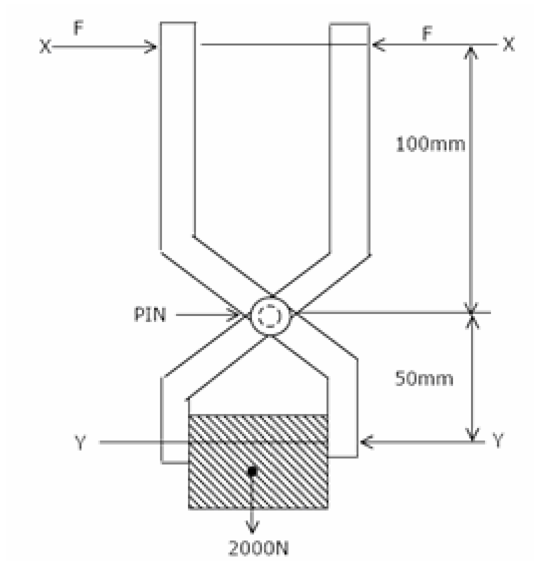
GATE-14. The figure shows a pair of pin-jointed gripper-tongs holding an object weighing 2000 N. The co-efficient of friction (μ) at the gripping surface is 0.1 XX is the line of action of the input force and YY is the line of application of gripping force. If the pin-joint is assumed to be frictionless, then magnitude of force F required to hold the weight is:

(a) 1000 N

(b) 2000 N

(c) 2500 N

(d) 5000 N



[GATE-2004]

GATE-14. Ans. (d) Frictional force required = 2000 N

Force needed to produce 2000N frictional force at Y-Y section = $\frac{2000}{0.1} = 20000\text{N}$

So for each side we need (F_y) = 10000 N force

Taking moment about PIN

$$F_y \times 50 = F \times 100 \quad \text{or } F = \frac{F_y \times 50}{100} = \frac{10000 \times 50}{100} = 5000\text{N}$$

GATE-15. A uniform, slender cylindrical rod is made of a homogeneous and isotropic material. The rod rests on a frictionless surface. The rod is heated uniformly. If the radial and longitudinal thermal stresses are represented by σ_r and σ_z, respectively, then [GATE-2005]

$(a)\sigma_r = 0, \sigma_z = 0$ $(b)\sigma_r \neq 0, \sigma_z = 0$ $(c)\sigma_r = 0, \sigma_z \neq 0$ $(d)\sigma_r \neq 0, \sigma_z \neq 0$

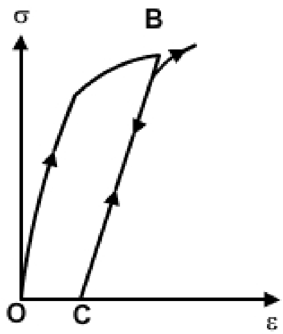
GATE-15. Ans. (a) Thermal stress will develop only when you prevent the material to contrast/elongate. As here it is free no thermal stress will develop.

Tensile Test

GATE-16. A test specimen is stressed slightly beyond the yield point and then unloaded. Its yield strength will [GATE-1995]

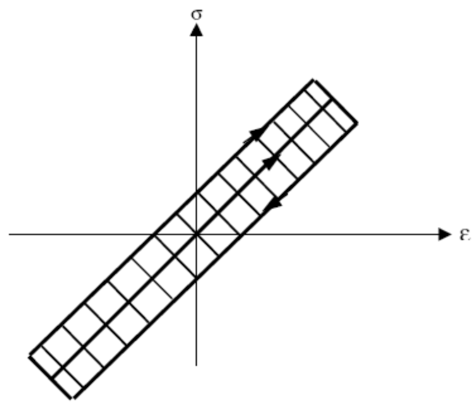
(a) Decrease (b) Increase
(c) Remains same (d) Becomes equal to ultimate tensile strength

GATE-16. Ans. (b)



GATE-17. Under repeated loading a material has the stress-strain curve shown in figure, which of the following statements is true?

(a) The smaller the shaded area, the better the material damping
(b) The larger the shaded area, the better the material damping
(c) Material damping is an independent material property and does not depend on this curve
(d) None of these



[GATE-1999]

GATE-17. Ans. (a)

Previous 20-Years IES Questions

Stress in a bar due to self-weight

IES-1. A solid uniform metal bar of diameter D and length L is hanging vertically from its upper end. The elongation of the bar due to self weight is: [IES-2005]

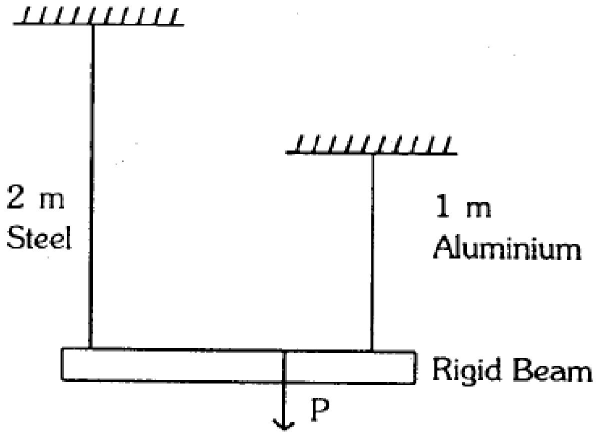
(a) Proportional to L and inversely proportional to D^2
(b) Proportional to L^2 and inversely proportional to D^2
(c) Proportional of L but independent of D
(d) Proportional of U but independent of D

IES-1. Ans. (a) $\delta = \dfrac{WL}{2AE} = \dfrac{WL}{2 \times \dfrac{\pi D^2}{4} \times E} \quad \therefore \delta \times L \qquad \& \quad \delta \times \dfrac{1}{D^2}$

IES-2. The deformation of a bar under its own weight as compared to that when subjected to a direct axial load equal to its own weight will be: [IES-1998]
(a) The same (b) One-fourth (c) Half (d) Double

IES-2. Ans. (c)

IES-3. A rigid beam of negligible weight is supported in a horizontal position by two rods of steel and aluminum, 2 m and 1 m long having values of cross - sectional areas 1 cm² and 2 cm² and E of 200 GPa and 100 GPa respectively. A load P is applied as shown in the figure [IES-2002]



If the rigid beam is to remain horizontal then
(a) The forces on both sides should be equal
(b) The force on aluminum rod should be twice the force on steel
(c) The force on the steel rod should be twice the force on aluminum
(d) The force P must be applied at the centre of the beam

IES-3. Ans. (b)

Bar of uniform strength

IES-4. Which one of the following statements is correct? [IES 2007]
A beam is said to be of uniform strength, if
(a) The bending moment is the same throughout the beam
(b) The shear stress is the same throughout the beam
(c) The deflection is the same throughout the beam
(d) The bending stress is the same at every section along its longitudinal axis

IES-4. Ans. (d)

IES-5. Which one of the following statements is correct? [IES-2006]
Beams of uniform strength vary in section such that
(a) bending moment remains constant (b) deflection remains constant
(c) maximum bending stress remains constant (d) shear force remains constant

IES-5. Ans. (c)

IES-6. For bolts of uniform strength, the shank diameter is made equal to [IES-2003]
(a) Major diameter of threads (b) Pitch diameter of threads
(c) Minor diameter of threads (d) Nominal diameter of threads

IES-6. Ans. (c)

IES-7. A bolt of uniform strength can be developed by [IES-1995]
(a) Keeping the core diameter of threads equal to the diameter of unthreaded portion of the bolt

(b) Keeping the core diameter smaller than the diameter of the unthreaded portion

(c) Keeping the nominal diameter of threads equal the diameter of unthreaded portion of the bolt

(d) One end fixed and the other end free

IES-7. Ans. (a)

Elongation of a Taper Rod

IES-8. Two tapering bars of the same material are subjected to a tensile load P. The lengths of both the bars are the same. The larger diameter of each of the bars is D. The diameter of the bar A at its smaller end is D/2 and that of the bar B is D/3. What is the ratio of elongation of the bar A to that of the bar B? [IES-2006]

(a) 3 : 2 (b) 2: 3 (c) 4 : 9 (d) 1: 3

IES-8. Ans. (b) Elongation of a taper rod $(\delta l) = \dfrac{PL}{\frac{\pi}{4} d_1 d_2 E}$

$$\text{or } \frac{(\delta l)_A}{(\delta l)_B} = \frac{(d_2)_B}{(d_2)_A} = \left(\frac{D/3}{D/2}\right) = \frac{2}{3}$$

IES-9. A bar of length L tapers uniformly from diameter 1.1 D at one end to 0.9 D at the other end. The elongation due to axial pull is computed using mean diameter D. What is the approximate error in computed elongation? [IES-2004]

(a) 10% (b) 5% (c) 1% (d) 0.5%

IES-9. Ans. (c) Actual elongation of the bar $(\delta l)_{act} = \dfrac{PL}{\left(\frac{\pi}{4} d_1 d_2\right)E} = \dfrac{PL}{\left(\frac{\pi}{4} \times 1.1D \times 0.9D\right)E}$

$$\text{Calculated elongation of the bar } (\delta l)_{Cal} = \frac{PL}{\frac{\pi D^2}{4} \times E}$$

$$\therefore \text{Error}(\%) = \frac{(\delta l)_{act} - (\delta l)_{cal}}{(\delta l)_{cal}} \times 100 = \left(\frac{D^2}{1.1D \times 0.9D} - 1\right) \times 100\% = 1\%$$

IES-10. The stretch in a steel rod of circular section, having a length 'l' subjected to a tensile load' P' and tapering uniformly from a diameter d_1 at one end to a diameter d_2 at the other end, is given [IES-1995]

(a) $\dfrac{Pl}{4Ed_1d_2}$ (b) $\dfrac{pl.\pi}{Ed_1d_2}$ (c) $\dfrac{pl.\pi}{4Ed_1d_2}$ (d) $\dfrac{4pl}{\pi Ed_1d_2}$

IES-10. Ans. (d) Actual elongation of the bar $(\delta l)_{act} = \dfrac{PL}{\left(\frac{\pi}{4} d_1 d_2\right)E}$

IES-11. A tapering bar (diameters of end sections being d_1 and d_2 a bar of uniform cross-section 'd' have the same length and are subjected the same axial pull. Both the bars will have the same extension if'd' is equal to [IES-1998]

(a) $\dfrac{d_1 + d_2}{2}$ (b) $\sqrt{d_1 d_2}$ (c) $\sqrt{\dfrac{d_1 d_2}{2}}$ (d) $\sqrt{\dfrac{d_1 + d_2}{2}}$

IES-11. Ans. (b)

Poisson's ratio

IES-12. In the case of an engineering material under unidirectional stress in the x-direction, the Poisson's ratio is equal to (symbols have the usual meanings) [IAS 1994, IES-2000]

(a) $\dfrac{\varepsilon_y}{\varepsilon_x}$ (b) $\dfrac{\varepsilon_y}{\sigma_x}$ (c) $\dfrac{\sigma_y}{\sigma_x}$ (d) $\dfrac{\sigma_y}{\varepsilon_x}$

IES-12. Ans. (a)

IES-13. Which one of the following is correct in respect of Poisson's ratio (v) limits for an isotropic elastic solid? [IES-2004]

(a) $-\infty \le \nu \le \infty$ (b) $1/4 \le \nu \le 1/3$ (c) $-1 \le \nu \le 1/2$ (d) $-1/2 \le \nu \le 1/2$

IES-13. Ans. (c) Theoretically $-1 < \mu < 1/2$ but practically $0 < \mu < 1/2$

IES-14. Match List-I (Elastic properties of an isotropic elastic material) with List-II (Nature of strain produced) and select the correct answer using the codes given below the Lists: [IES-1997]

List-I	List-II
A. Young's modulus	**1.** Shear strain
B. Modulus of rigidity	**2.** Normal strain
C. Bulk modulus	**3.** Transverse strain
D. Poisson's ratio	**4.** Volumetric strain

Codes:	A	B	C	D		A	B	C	D
(a)	1	2	3	4	(b)	2	1	3	4
(c)	2	1	4	3	(d)	1	2	4	3

IES-14. Ans. (c)

IES-15. If the value of Poisson's ratio is zero, then it means that [IES-1994]

(a) The material is rigid.
(b) The material is perfectly plastic.
(c) There is no longitudinal strain in the material
(d) The longitudinal strain in the material is infinite.

IES-15. Ans. (a) If Poisson's ratio is zero, then material is rigid.

IES-16. Which of the following is true (μ= Poisson's ratio) [IES-1992]

(a) $0 < \mu < 1/2$ (b) $1 < \mu < 0$ (c) $1 < \mu < -1$ (d) $\infty < \mu << -\infty$

IES-16. Ans. (a)

Elasticity and Plasticity

IES-17. If the area of cross-section of a wire is circular and if the radius of this circle decreases to half its original value due to the stretch of the wire by a load, then the modulus of elasticity of the wire be: [IES-1993]

(a) One-fourth of its original value (b) Halved (c) Doubled (d) Unaffected

IES-17. Ans. (d) Note: Modulus of elasticity is the property of material. It will remain same.

IES-18. The relationship between the Lame's constant 'λ', Young's modulus 'E' and the Poisson's ratio 'μ' [IES-1997]

(a) $\lambda = \dfrac{E\mu}{(1+\mu)(1-2\mu)}$ (b) $\lambda = \dfrac{E\mu}{(1+2\mu)(1-\mu)}$ (c) $\lambda = \dfrac{E\mu}{1+\mu}$ (d) $\lambda = \dfrac{E\mu}{(1-\mu)}$

IES-18. Ans. (a)

IES-19. Which of the following pairs are correctly matched? [IES-1994]

1. Resilience............... Resistance to deformation.
2. Malleability ....Shape change.
3. Creep .... Progressive deformation.
4. Plasticity ....Permanent deformation.

Select the correct answer using the codes given below:

Codes: (a) 2, 3 and 4 (b) 1, 2 and 3 (c) 1, 2 and 4 (d) 1, 3 and 4

IES-19. Ans. (a) Strain energy stored by a body within elastic limit is known as resilience.

Creep and fatigue

IES-20. What is the phenomenon of progressive extension of the material i.e., strain increasing with the time at a constant load, called? [IES 2007]
(a) Plasticity (b) Yielding (b) Creeping (d) Breaking

IES-20. Ans. (c)

IES-21. The correct sequence of creep deformation in a creep curve in order of their elongation is: [IES-2001]
(a) Steady state, transient, accelerated (b) Transient, steady state, accelerated
(c) Transient, accelerated, steady state (d) Accelerated, steady state, transient

IES-21. Ans. (b)

IES-22. The highest stress that a material can withstand for a specified length of time without excessive deformation is called [IES-1997]
(a) Fatigue strength (b) Endurance strength
(c) Creep strength (d) Creep rupture strength

IES-22. Ans. (c)

IES-23. Which one of the following features improves the fatigue strength of a metallic material? [IES-2000]
(a) Increasing the temperature (b) Scratching the surface
(c) Overstressing (d) Under stressing

IES-23. Ans. (d)

IES-24. Consider the following statements: [IES-1993]
For increasing the fatigue strength of welded joints it is necessary to employ
1. Grinding 2. Coating 3. Hammer peening
Of the above statements
(a) 1 and 2 are correct (b) 2 and 3 are correct
(c) 1 and 3 are correct (d) 1, 2 and 3 are correct

IES-24. Ans. (c) A polished surface by grinding can take more number of cycles than a part with rough surface. In Hammer peening residual compressive stress lower the peak tensile stress

Relation between the Elastic Modulii

IES-25. For a linearly elastic, isotropic and homogeneous material, the number of elastic constants required to relate stress and strain is: [IAS 1994; IES-1998]
(a) Two (b) Three (c) Four (d) Six

IES-25. Ans. (a)

IES-26. E, G, K and μ represent the elastic modulus, shear modulus, bulk modulus and Poisson's ratio respectively of a linearly elastic, isotropic and homogeneous material. To express the stress-strain relations completely for this material, at least [IES-2006]
(a) E, G and μ must be known (b) E, K and μ must be known
(c) Any two of the four must be known (d) All the four must be known

IES-26. Ans. (c)

IES-27. The number of elastic constants for a completely anisotropic elastic material which follows Hooke's law is: [IES-1999]
(a) 3 (b) 4 (c) 21 (d) 25

IES-27. Ans. (c)

IES-28. What are the materials which show direction dependent properties, called?
(a) Homogeneous materials (b) Viscoelastic materials **[IES 2007]**
(c) Isotropic materials (d) Anisotropic materials

IES-28. Ans. (d)

IES-29. An orthotropic material, under plane stress condition will have: [IES-2006]

(a) 15 independent elastic constants (b) 4 independent elastic constants
(c) 5 independent elastic constants (d) 9 independent elastic constants

IES-29. Ans. (d)

IES-30. Match List-I (Properties) with List-II (Units) and select the correct answer using the codes given below the lists: [IES-2001]

List I	**List II**
A. Dynamic viscosity	**1.** Pa
B. Kinematic viscosity	**2.** m^2/s
C. Torsional stiffness	**3.** Ns/m^2
D. Modulus of rigidity	**4.** N/m

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	3	2	4	1	(b)	5	2	4	3
(b)	3	4	2	3	(d)	5	4	2	1

IES-30. Ans. (a)

IES-31. Young's modulus of elasticity and Poisson's ratio of a material are 1.25 × 10^5 MPa and 0.34 respectively. The modulus of rigidity of the material is: [IAS 1994, IES-1995, 2001, 2002, 2007]

(a) 0.4025 ×10^5 Mpa (b) 0.4664 × 10^5 Mpa
(c) 0.8375 × 10^5 MPa (d) 0.9469 × 10^5 MPa

IES-31. Ans.(b) $E = 2G(1+\mu)$ or 1.25x10^5 = 2G(1+0.34) or G = 0.4664 × 10^5 MPa

IES-32. In a homogenous, isotropic elastic material, the modulus of elasticity E in terms of G and K is equal to [IAS-1995, IES - 1992]

(a) $\frac{G+3K}{9KG}$ (b) $\frac{3G+K}{9KG}$ (c) $\frac{9KG}{G+3K}$ (d) $\frac{9KG}{K+3G}$

IES-32. Ans. (c)

IES-33. What is the relationship between the linear elastic properties Young's modulus (E), rigidity modulus (G) and bulk modulus (K)? [IES-2008]

(a) $\frac{1}{E}=\frac{9}{K}+\frac{3}{G}$ (b) $\frac{3}{E}=\frac{9}{K}+\frac{1}{G}$ (c) $\frac{9}{E}=\frac{3}{K}+\frac{1}{G}$ (d) $\frac{9}{E}=\frac{1}{K}+\frac{3}{G}$

IES-33. Ans. (d) $E = 2G(1+\mu) = 3K(1-2\mu) = \frac{9KG}{3K+G}$

IES-34. What is the relationship between the liner elastic properties Young's modulus (E), rigidity modulus (G) and bulk modulus (K)? [IES-2009]

(a) $E=\frac{KG}{9K+G}$ (b) $E=\frac{9KG}{K+G}$ (c) $E=\frac{9KG}{K+3G}$ (d) $E=\frac{9KG}{3K+G}$

IES-34. Ans. (d) $E = 2G(1+\mu) = 3K(1-2\mu) = \frac{9KG}{3K+G}$

IES-35. If E, G and K denote Young's modulus, Modulus of rigidity and Bulk Modulus, respectively, for an elastic material, then which one of the following can be possibly true? [IES-2005]

(a) G = 2K (b) G = E (c) K = E (d) G = K = E

IES-35. Ans.(c) $E = 2G(1+\mu) = 3K(1-2\mu) = \frac{9KG}{3K+G}$

the value of μ must be between 0 to 0.5 so E never equal to G but if $\mu = \frac{1}{3}$ then

$E = k$ so ans. is c

IES-36. If a material had a modulus of elasticity of 2.1 × 10^6 kgf/cm^2 and a modulus of rigidity of 0.8 × 10^6 kgf/cm^2 then the approximate value of the Poisson's ratio of the material would be: [IES-1993]

(a) 0.26 (b) 0.31 (c) 0.47 (d) 0.5

IES-36. Ans. (b) Use $E = 2G(1+\mu)$

IES-37. The modulus of elasticity for a material is 200 GN/m² and Poisson's ratio is 0.25. What is the modulus of rigidity? [IES-2004]

(a) 80 GN/m² (b) 125 GN/m² (c) 250 GN/m² (d) 320 GN/m²

IES-37. Ans. (a) $E = 2G(1+\mu)$ or $G = \frac{E}{2(1+\mu)} = \frac{200}{2\times(1+0.25)} = 80GN/m^2$

IES-38. Consider the following statements: [IES-2009]

1. **Two-dimensional stresses applied to a thin plate in its own plane represent the plane stress condition.**
2. **Under plane stress condition, the strain in the direction perpendicular to the plane is zero.**
3. **Normal and shear stresses may occur simultaneously on a plane.**

Which of the above statements is /are correct?

(a) 1 only (b) 1 and *2* (c) 2 and 3 (d) 1 and 3

IES-38. Ans. (d) Under plane stress condition, the strain in the direction perpendicular to the plane is not zero. It has been found experimentally that when a body is stressed within elastic limit, the lateral strain bears a constant ratio to the linear strain. **[IES-2009]**

Stresses in compound strut

IES-39. Eight bolts are to be selected for fixing the cover plate of a cylinder subjected to a maximum load of 980·175 kN. If the design stress for the bolt material is 315 N/mm², what is the diameter of each bolt? [IES-2008]

(a) 10 mm (b) 22 mm (c) 30 mm (d) 36 mm

IES-39. Ans. (b) $\text{Total load}(P) = 8\times\sigma\times\frac{\pi d^2}{4}$ or $d = \sqrt{\frac{P}{2\pi\sigma}} = \sqrt{\frac{980175}{2\pi\times 315}} = 22.25\text{mm}$

IES-40. For a composite consisting of a bar enclosed inside a tube of another material when compressed under a load 'w' as a whole through rigid collars at the end of the bar. The equation of compatibility is given by (suffixes 1 and 2) refer to bar and tube respectively [IES-1998]

$(a) W_1 + W_2 = W$ $(b) W_1 + W_2 = Const.$ $(c) \frac{W_1}{A_1E_1} = \frac{W_2}{A_2E_2}$ $(d) \frac{W_1}{A_1E_2} = \frac{W_2}{A_2E_1}$

IES-40. Ans. (c) Compatibility equation insists that the change in length of the bar must be compatible with the boundary conditions. Here (a) is also correct but it is equilibrium equation.

IES-41. When a composite unit consisting of a steel rod surrounded by a cast iron tube is subjected to an axial load. [IES-2000]

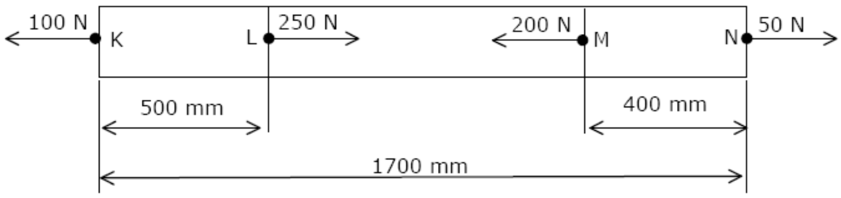
Assertion (A): The ratio of normal stresses induced in both the materials is equal to the ratio of Young's moduli of respective materials.

Reason (R): The composite unit of these two materials is firmly fastened together at the ends to ensure equal deformation in both the materials.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **not** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-41. Ans. (a)

IES-42. The figure below shows a steel rod of 25 mm² cross sectional area. It is loaded at four points, K, L, M and N. [GATE-2004, IES 1995, 1997, 1998]

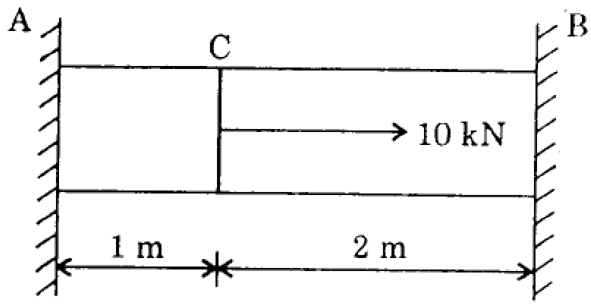


Assume E_{steel} = 200 GPa. The total change in length of the rod due to loading is
(a) 1 μm (b) -10 μm (c) 16 μm (d) -20 μm

IES-42. Ans. (b) First draw FBD of all parts separately then

Total change in length = $\sum \frac{PL}{AE}$

IES-43. The reactions at the rigid supports at A and B for the bar loaded as shown in the figure are respectively.
(a) 20/3 kN,10/3 kN
(b) 10/3 kN, 20/3 kN
(c) 5 kN, 5 kN
(d) 6 kN, 4 kN



[IES-2002; IAS-2003]

IES-43. Ans. (a) Elongation in AC = length reduction in CB

$$\frac{R_A \times 1}{AE} = \frac{R_B \times 2}{AE}$$

And $R_A + R_B = 10$

IES-44. Which one of the following is correct? [IES-2008]
When a nut is tightened by placing a washer below it, the bolt will be subjected to
(a) Compression only (b) Tension
(c) Shear only (d) Compression and shear

IES-44. Ans. (b)

IES-45. Which of the following stresses are associated with the tightening of nut on a bolt? [IES-1998]
1. **Tensile stress due to the stretching of bolt**
2. **Bending stress due to the bending of bolt**
3. **Crushing and shear stresses in threads**
4. **Torsional shear stress due to frictional resistance between the nut and the bolt.**

Select the correct answer using the codes given below
Codes: (a) 1, 2 and 4 (b) 1, 2 and 3 (c) 2, 3 and 4 (d) 1, 3 and 4

IES-45. Ans. (d)

Thermal effect

IES-46. A 100 mm × 5 mm × 5 mm steel bar free to expand is heated from 15°C to 40°C. What shall be developed? [IES-2008]
(a) Tensile stress (b) Compressive stress (c) Shear stress (d) No stress

IES-46. Ans. (d) If we resist to expand then only stress will develop.

IES-47. Which one of the following statements is correct? [GATE-1995; IES 2007]
If a material expands freely due to heating, it will develop

(a) Thermal stress (b) Tensile stress (c) Compressive stress (d) No stress

IES-47. Ans. (d)

IES-48. A cube having each side of length a, is constrained in all directions and is heated uniformly so that the temperature is raised to T°C. If α is the thermal coefficient of expansion of the cube material and E the modulus of elasticity, the stress developed in the cube is: [IES-2003]

(a) $\frac{\alpha TE}{\gamma}$ (b) $\frac{\alpha TE}{(1-2\gamma)}$ (c) $\frac{\alpha TE}{2\gamma}$ (d) $\frac{\alpha TE}{(1+2\gamma)}$

IES-48. Ans. (b) $\frac{\Delta V}{V} = \frac{\sigma = (p)}{K} = \frac{a^3(1+\alpha T)^3 - a^3}{a^3}$

$$Or \frac{P}{\frac{E}{3(1-2\gamma)}} = 3\alpha T$$

IES-49. Consider the following statements: [IES-2002]
Thermal stress is induced in a component in general, when
1. **A temperature gradient exists in the component**
2. **The component is free from any restraint**
3. **It is restrained to expand or contract freely**

Which of the above statements are correct?

(a) 1 and 2 (b) 2 and 3 (c) 3 alone (d) 2 alone

IES-49. Ans. (c)

IES-50. A steel rod 10 mm in diameter and 1m long is heated from 20°C to 120°C, E = 200 GPa and α = 12 × 10^{-6} per °C. If the rod is not free to expand, the thermal stress developed is: [IAS-2003, IES-1997, 2000, 2006]

(a) 120 MPa (tensile) (b) 240 MPa (tensile)
(c) 120 MPa (compressive) (d) 240 MPa (compressive)

IES-50. Ans. (d) $\alpha E\Delta t = (12\times10^{-6})\times(200\times10^3)\times(120-20) = 240\,\text{MPa}$

It will be compressive as elongation restricted.

IES-51. A cube with a side length of 1 cm is heated uniformly 1° C above the room temperature and all the sides are free to expand. What will be the increase in volume of the cube? (Given coefficient of thermal expansion is α per °C)

(a) 3 α cm^3 (b) 2 α cm^3 (c) α cm^3 (d) zero **[IES-2004]**

IES-51. Ans. (a) co-efficient of volume expansion $(\gamma) = 3\times$ co – efficient of linear expansion (α)

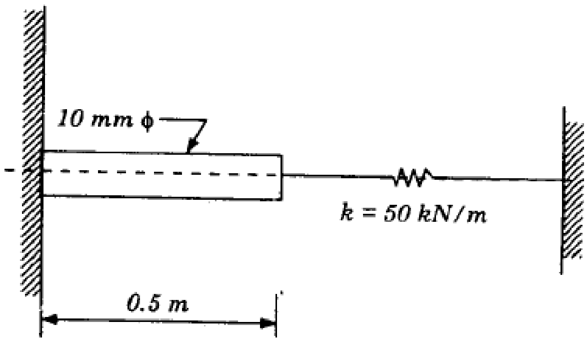
IES-52. A bar of copper and steel form a composite system. [IES-2004]
They are heated to a temperature of 40 ° C. What type of stress is induced in the copper bar?

(a) Tensile (b) Compressive (c) Both tensile and compressive (d) Shear

IES-52. Ans. (b)

IES-53. $\alpha = 12.5\times10^{-6}/^\circ C,\quad E = 200\,GPa$ **If the rod fitted strongly between the supports as shown in the figure, is heated, the stress induced in it due to 20°C rise in temperature will be: [IES-1999]**

(a) 0.07945 MPa (b) -0.07945 MPa (c) -0.03972 MPa (d) 0.03972 MPa



IES-53. Ans. (b) Let compression of the spring = x m

Therefore spring force = kx kN

Expansion of the rod due to temperature rise = $L\alpha\Delta t$

Reduction in the length due to compression force = $\dfrac{(kx)\times L}{AE}$

Now $L\alpha\Delta t - \dfrac{(kx)\times L}{AE} = x$

Or $x = \dfrac{0.5\times 12.5\times 10^{-6}\times 20}{\left\{1+\dfrac{50\times 0.5}{\dfrac{\pi\times 0.010^2}{4}\times 200\times 10^6}\right\}} = 0.125\,\text{mm}$

$\therefore$ Compressive stress = $-\dfrac{kx}{A} = -\dfrac{50\times 0.125}{\left(\dfrac{\pi\times 0.010^2}{4}\right)} = -0.07945\,\text{MPa}$

IES-54. The temperature stress is a function of [IES-1992]
1. Coefficient of linear expansion 2. Temperature rise 3. Modulus of elasticity
The correct answer is:
(a) 1 and 2 only (b) 1 and 3 only (c) 2 and 3 only (d) 1, 2 and 3

IES-54. Ans. (d) Stress in the rod due to temperature rise = $(\alpha\Delta t)\times E$

Impact loading

IES-55. Assertion (A): Ductile materials generally absorb more impact loading than a brittle material [IES-2004]
Reason (R): Ductile materials generally have higher ultimate strength than brittle materials
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **not** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-55. Ans. (c)

IES-56. Assertion (A): Specimens for impact testing are never notched. [IES-1999]
Reason (R): A notch introduces tri-axial tensile stresses which cause brittle fracture.
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-56. Ans. (d) A is false but R is correct.

Tensile Test

IES-57. During tensile-testing of a specimen using a Universal Testing Machine, the parameters actually measured include [IES-1996]

(a) True stress and true strain (b) Poisson's ratio and Young's modulus
(c) Engineering stress and engineering strain (d) Load and elongation

IES-57. Ans. (d)

IES-58. In a tensile test, near the elastic limit zone [IES-2006]

(a) Tensile stress increases at a faster rate
(b) Tensile stress decreases at a faster rate
(c) Tensile stress increases in linear proportion to the stress
(d) Tensile stress decreases in linear proportion to the stress

IES-58. Ans. (b)

IES-59. Match List-I (Types of Tests and Materials) with List-II (Types of Fractures) and select the correct answer using the codes given below the lists:

List I (Types of Tests and Materials)	**List-II (Types of Fractures)** [IES-2002; IAS-2004]
A. Tensile test on CI	**1.** Plain fracture on a transverse plane
B. Torsion test on MS	**2.** Granular helecoidal fracture
C. Tensile test on MS	**3.** Plain granular at 45° to the axis
D. Torsion test on CI	**4.** Cup and Cone
	5. Granular fracture on a transverse plane

Codes:

	A	B	C	D		A	B	C	D
(a)	4	2	3	1	(c)	4	1	3	2
(b)	5	1	4	2	(d)	5	2	4	1

IES-59. Ans. (d)

IES-60. Which of the following materials generally exhibits a yield point? [IES-2003]

(a) Cast iron (b) Annealed and hot-rolled mild steel
(c) Soft brass (d) Cold-rolled steel

IES-60. Ans. (b)

IES-61. For most brittle materials, the ultimate strength in compression is much large then the ultimate strength in tension. The is mainly due to [IES-1992]

(a) Presence of flaws and microscopic cracks or cavities
(b) Necking in tension
(c) Severity of tensile stress as compared to compressive stress
(d) Non-linearity of stress-strain diagram

IES-61. Ans. (a)

IES-62. What is the safe static tensile load for a M36 × 4C bolt of mild steel having yield stress of 280 MPa and a factor of safety 1.5? [IES-2005]

(a) 285 kN (b) 190 kN (c) 142.5 kN (d) 95 kN

IES-62. Ans. (b) $\sigma_c = \dfrac{W}{\dfrac{\pi d^2}{4}}$ or $W = \sigma_c \times \dfrac{\pi d^2}{4}$;

$$W_{safe} = \frac{W}{fos} = \frac{\sigma_c \times \pi \times d^2}{fos \times 4} = \frac{280 \times \pi \times 36^2}{1.5 \times 4} N = 190 kN$$

IES-63. Which one of the following properties is more sensitive to increase in strain rate? [IES-2000]

(a) Yield strength (b) Proportional limit (c) Elastic limit (d) Tensile strength

IES-63. Ans. (b)

IES-64. A steel hub of 100 mm internal diameter and uniform thickness of 10 mm was heated to a temperature of 300°C to shrink-fit it on a shaft. On cooling, a crack developed parallel to the direction of the length of the hub. Consider the following factors in this regard: [IES-1994]

1. Tensile hoop stress **2. Tensile radial stress**
3. Compressive hoop stress **4. Compressive radial stress**

The cause of failure is attributable to

(a) 1 alone (b) 1 and 3 (c) 1, 2 and 4 (d) 2, 3 and 4

IES-64. Ans. (a) A crack parallel to the direction of length of hub means the failure was due to tensile hoop stress only.

IES-65. If failure in shear along 45° planes is to be avoided, then a material subjected to uniaxial tension should have its shear strength equal to at least [IES-1994]

(a) Tensile strength (b) Compressive strength
(c) Half the difference between the tensile and compressive strengths.
(d) Half the tensile strength.

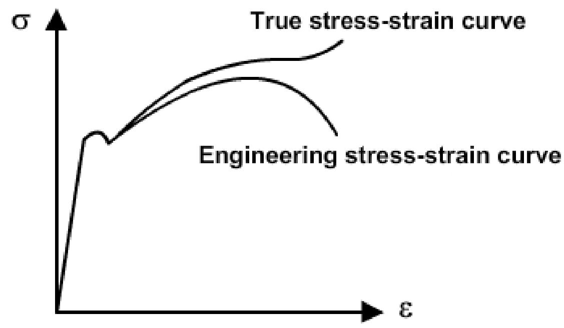
IES-65. Ans. (d)

IES-66. Select the proper sequence [IES-1992]

1. Proportional Limit **2. Elastic limit** **3. Yielding** **4. Failure**

(a) 2, 3, 1, 4 (b) 2, 1, 3, 4 (c) 1, 3, 2, 4 (d) 1, 2, 3, 4

IES-66. Ans. (d)



Previous 20-Years IAS Questions

Stress in a bar due to self-weight

IAS-1. A heavy uniform rod of length 'L' and material density 'δ' is hung vertically with its top end rigidly fixed. How is the total elongation of the bar under its own weight expressed? [IAS-2007]

(a) $\dfrac{2\delta L^2 g}{E}$ (b) $\dfrac{\delta L^2 g}{E}$ (c) $\dfrac{\delta L^2 g}{\sqrt{2}E}$ (d) $\dfrac{\delta L^2 g}{2E}$

IAS-1. Ans. (d) Elongation due to self weight $=\dfrac{WL}{2AE}=\dfrac{(\delta ALg)L}{2AE}=\dfrac{\delta L^2 g}{2E}$

IAS-2. A rod of length '*l*' and cross-section area 'A' rotates about an axis passing through one end of the rod. The extension produced in the rod due to centrifugal forces is (w is the weight of the rod per unit length and ω is the angular velocity of rotation of the rod). [IAS 1994]

(a) $\dfrac{\omega w l^2}{gE}$ (b) $\dfrac{\omega^2 w l^3}{3gE}$ (c) $\dfrac{\omega^2 w l^3}{gE}$ (d) $\dfrac{3gE}{\omega^2 w l^3}$

IAS-2. Ans. (b)

Elongation of a Taper Rod

IAS-3. A rod of length, "l" tapers uniformly from a diameter "D_1' to a diameter "D_2' and carries an axial tensile load of "P". The extension of the rod is (E represents the modulus of elasticity of the material of the rod) [IAS-1996]

(a) $\frac{4Pl}{\pi ED_1D_2}$ (b) $\frac{4PEl}{\pi D_1D_2}$ (c) $\frac{\pi EPl}{4D_1D_2}$ (d) $\frac{\pi Pl}{4ED_1D_2}$

IAS-3. Ans. (a) The extension of the taper rod = $\frac{Pl}{\left(\frac{\pi}{4}D_1D_2\right).E}$

Poisson's ratio

IAS-4. In the case of an engineering material under unidirectional stress in the x-direction, the Poisson's ratio is equal to (symbols have the usual meanings) [IAS 1994, IES-2000]

(a) $\frac{\varepsilon_y}{\varepsilon_x}$ (b) $\frac{\varepsilon_y}{\sigma_x}$ (c) $\frac{\sigma_y}{\sigma_x}$ (d) $\frac{\sigma_y}{\varepsilon_x}$

IAS-4. Ans. (a)

IAS-5. Assertion (A): Poisson's ratio of a material is a measure of its ductility.
Reason (R): For every linear strain in the direction of force, Poisson's ratio of the material gives the lateral strain in directions perpendicular to the direction of force. [IAS-1999]

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **not** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-5. ans. (d)

IAS-6. Assertion (A): Poisson's ratio is a measure of the lateral strain in all direction perpendicular to and in terms of the linear strain. [IAS-1997]
Reason (R): The nature of lateral strain in a uni-axially loaded bar is opposite to that of the linear strain.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **not** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-6. Ans. (b)

Elasticity and Plasticity

IAS-7. A weight falls on a plunger fitted in a container filled with oil thereby producing a pressure of 1.5 N/mm^2 in the oil. The Bulk Modulus of oil is 2800 N/mm^2. Given this situation, the volumetric compressive strain produced in the oil will be: [IAS-1997]

(a) 400×10^{-6} (b) 800×10^{6} (c) 268×10^{6} (d) 535×10^{-6}

IAS-7. Ans. (d) Bulk modulus of elasticity (K) = $\frac{P}{\varepsilon_v}$ or $\varepsilon_v = \frac{P}{K} = \frac{1.5}{2800} = 535 \times 10^{-6}$

Relation between the Elastic Modulii

IAS-8. For a linearly elastic, isotropic and homogeneous material, the number of elastic constants required to relate stress and strain is: [IAS 1994; IES-1998]

(a) Two (b) Three (c) Four (d) Six

IAS-8. Ans. (a)

IAS-9. The independent elastic constants for a homogenous and isotropic material are
(a) E, G, K, v (b) E, G, K (c) E, G, v (d) E, G **[IAS-1995]**

IAS-9. Ans. (d)

IAS-10. The unit of elastic modulus is the same as those of [IAS 1994]
(a) Stress, shear modulus and pressure (b) Strain, shear modulus and force
(c) Shear modulus, stress and force (d) Stress, strain and pressure.

IAS-10. Ans. (a)

IAS-11. Young's modulus of elasticity and Poisson's ratio of a material are 1.25 × 10^5 MPa and 0.34 respectively. The modulus of rigidity of the material is: [IAS 1994, IES-1995, 2001, 2002, 2007]
(a) 0.4025 × 10^5 MPa (b) 0.4664 × 10^5 MPa
(c) 0.8375 × 10^5 MPa (d) 0.9469 × 10^5 MPa

IAS-11. Ans.(b) $E = 2G(1+\mu)$ or 1.25x10^5 = 2G(1+0.34) or G = 0.4664 × 10^5 MPa

IAS-12. The Young's modulus of elasticity of a material is 2.5 times its modulus of rigidity. The Posson's ratio for the material will be: [IAS-1997]
(a) 0.25 (b) 0.33 (c) 0.50 (d) 0.75

IAS-12. Ans. (a) $E = 2G(1+\mu) \quad \Rightarrow 1+\mu = \frac{E}{2G} \quad \Rightarrow \mu = \left(\frac{E}{2G}-1\right) = \left(\frac{2.5}{2}-1\right) = 0.25$

IAS-13. In a homogenous, isotropic elastic material, the modulus of elasticity E in terms of G and K is equal to [IAS-1995, IES - 1992]
(a) $\frac{G+3K}{9KG}$ (b) $\frac{3G+K}{9KG}$ (c) $\frac{9KG}{G+3K}$ (d) $\frac{9KG}{K+3G}$

IAS-13. Ans. (c)

IAS-14. The Elastic Constants E and K are related as (μ is the Poisson's ratio) [IAS-1996]
(a) E = 2k (1 – 2 μ) (b) E = 3k (1- 2 μ) (c) E = 3k (1 + μ) (d) E = 2K(1 + 2 μ)

IAS-14. Ans. (b) E = 2G (1 + μ) = 3k (1- 2 μ)

IAS-15. For an isotropic, homogeneous and linearly elastic material, which obeys Hooke's law, the number of independent elastic constant is: [IAS-2000]
(a) 1 (b) 2 (c) 3 (d) 6

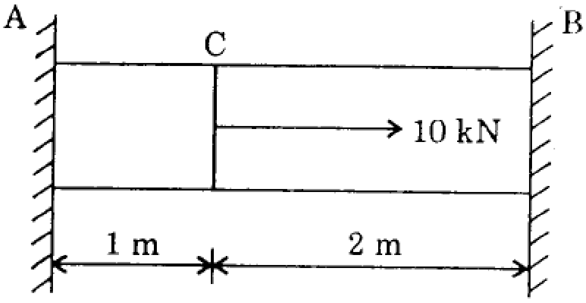
IAS-15. Ans. (b) E, G, K and μ represent the elastic modulus, shear modulus, bulk modulus and poisons ratio respectively of a 'linearly elastic, isotropic and homogeneous material.' To express the stress – strain relations completely for this material; at least **any two** of the four must be known. $E = 2G(1+\mu) = 3K(1-3\mu) = \frac{9KG}{3K+G}$

IAS-16. The moduli of elasticity and rigidity of a material are 200 GPa and 80 GPa, respectively. What is the value of the Poisson's ratio of the material? [IAS-2007]
(a) 0·30 (b) 0·26 (c) 0·25 (d) 0·24

IAS-16. Ans. (c) E = 2G (1+ μ) or $\mu = \frac{E}{2G} - 1 = \frac{200}{2\times 80} - 1 = 0.25$

Stresses in compound strut

IAS-17. The reactions at the rigid supports at A and B for the bar loaded as shown in the figure are respectively. [IES-2002; IAS-2003]
(a) 20/3 kN,10/3 Kn (b) 10/3 kN, 20/3 kN (c) 5 kN, 5 kN (d) 6 kN, 4 kN



IAS-17. Ans. (a) Elongation in AC = length reduction in CB

$$\frac{R_A \times 1}{AE} = \frac{R_B \times 2}{AE}$$

And $R_A + R_B = 10$

Thermal effect

IAS-18. A steel rod 10 mm in diameter and 1m long is heated from 20°C to 120°C, E = 200 GPa and α = 12 × 10^{-6} per °C. If the rod is not free to expand, the thermal stress developed is: [IAS-2003, IES-1997, 2000, 2006]

(a) 120 MPa (tensile) (b) 240 MPa (tensile)
(c) 120 MPa (compressive) (d) 240 MPa (compressive)

IAS-18. Ans. (d) $\alpha E\Delta t = \left(12\times10^{-6}\right)\times\left(200\times10^{3}\right)\times\left(120-20\right) = 240\,\text{MPa}$

It will be compressive as elongation restricted.

IAS-19. A. steel rod of diameter 1 cm and 1 m long is heated from 20°C to 120°C. Its $\alpha = 12\times10^{-6}/K$ and E=200 GN/m². If the rod is free to expand, the thermal stress developed in it is: [IAS-2002]

(a) 12 × 10^4 N/m² (b) 240 kN/m² (c) zero (d) infinity

IAS-19. Ans. (c) Thermal stress will develop only if expansion is restricted.

IAS-20. Which one of the following pairs is NOT correctly matched? [IAS-1999]
(E = Young's modulus, α = Coefficient of linear expansion, T = Temperature rise, A = Area of cross-section, l= Original length)

(a) Temperature strain with permitted expansion δ ..... $(\alpha Tl - \delta)$
(b) Temperature stress ..... αTE
(c) Temperature thrust ..... αTEA
(d) Temperature stress with permitted expansion ..... $\dfrac{E(\alpha Tl - \delta)}{l}$

IAS-20. Ans. (a) Dimensional analysis gives (a) is wrong

Impact loading

IAS-21. Match List I with List II and select the correct answer using the codes given below the lists: [IAS-1995]

List I (Property)	**List II (Testing Machine)**
A. Tensile strength	**1.** Rotating Bending Machine
B. Impact strength	**2.** Three-Point Loading Machine
C. Bending strength	**3.** Universal Testing Machine
D. Fatigue strength	**4.** Izod Testing Machine

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	4	3	2	1	(b)	3	2	1	4
(c)	2	1	4	3	(d)	3	4	2	1

IAS-21. Ans. (d)

Tensile Test

IAS-22. A mild steel specimen is tested in tension up to fracture in a Universal Testing Machine. Which of the following mechanical properties of the material can be evaluated from such a test? [IAS-2007]

1. Modulus of elasticity 2. Yield stress 3. Ductility
4. Tensile strength 5. Modulus of rigidity

Select the correct answer using the code given below:

(a) 1, 3, 5 and 6 (b) 2, 3, 4 and 6 (c) 1, 2, 5 and 6 (d) 1, 2, 3 and 4

IAS-22. Ans. (d)

IAS-23. In a simple tension test, Hooke's law is valid upto the [IAS-1998]

(a) Elastic limit (b) Limit of proportionality (c) Ultimate stress (d) Breaking point

IAS-23. Ans. (b)

IAS-24. Lueder' lines on steel specimen under simple tension test is a direct indication of yielding of material due to slip along the plane [IAS-1997]

(a) Of maximum principal stress (b) Off maximum shear
(c) Of loading (d) Perpendicular to the direction of loading

IAS-24. Ans. (b)

IAS-25. The percentage elongation of a material as obtained from static tension test depends upon the [IAS-1998]

(a) Diameter of the test specimen (b) Gauge length of the specimen
(c) Nature of end-grips of the testing machine (d) Geometry of the test specimen

IAS-25. Ans. (b)

IAS-26. Match List-I (Types of Tests and Materials) with List-II (Types of Fractures) and select the correct answer using the codes given below the lists:

List I (Types of Tests and Materials)	**List-II (Types of Fractures)** [IES-2002; IAS-2004]
A. Tensile test on CI	**1.** Plain fracture on a transverse plane
B. Torsion test on MS	**2.** Granular helecoidal fracture
C. Tensile test on MS	**3.** Plain granular at 45° to the axis
D. Torsion test on CI	**4.** Cup and Cone
	5. Granular fracture on a transverse plane

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	4	2	3	1	(c)	4	1	3	2
(b)	5	1	4	2	(d)	5	2	4	1

IAS-26. Ans. (d)

IAS-27. Assertion (A): For a ductile material stress-strain curve is a straight line up to the yield point. [IAS-2003]

Reason (R): The material follows Hooke's law up to the point of proportionality.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **not** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-27. Ans. (d)

IAS-28. Assertion (A): Stress-strain curves for brittle material do not exhibit yield point. [IAS-1996]

Reason (R): Brittle materials fail without yielding.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-28. Ans. (a) Up to elastic limit.

IAS-29. Match List I (Materials) with List II (Stress-Strain curves) and select the correct answer using the codes given below the Lists: **[IAS-2001]**

List I

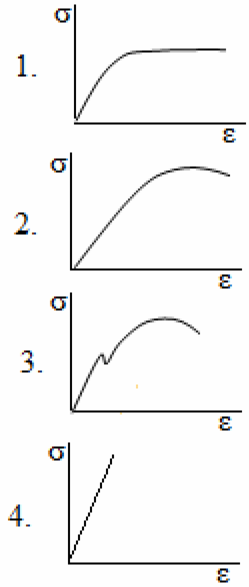
A. Mild Steel

B. Pure copper

C. Cast iron

D. Pure aluminium

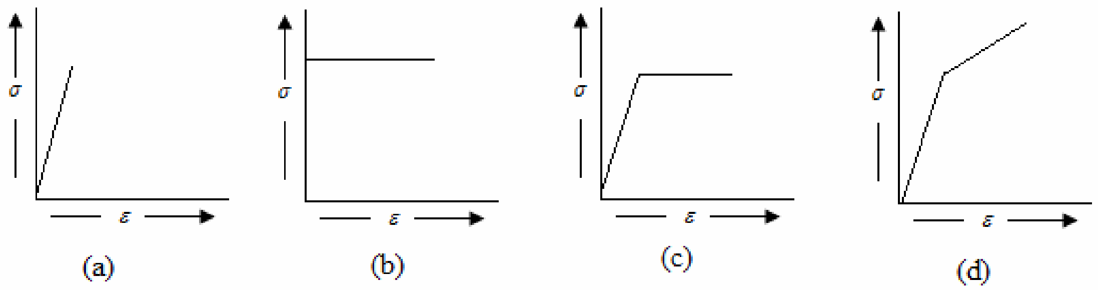
List II



Codes:	A	B	C	D		A	B	C	D
(a)	3	1	4	1	(b)	3	2	4	2
(c)	2	4	3	1	(d)	4	1	3	2

IAS-29. Ans. (b)

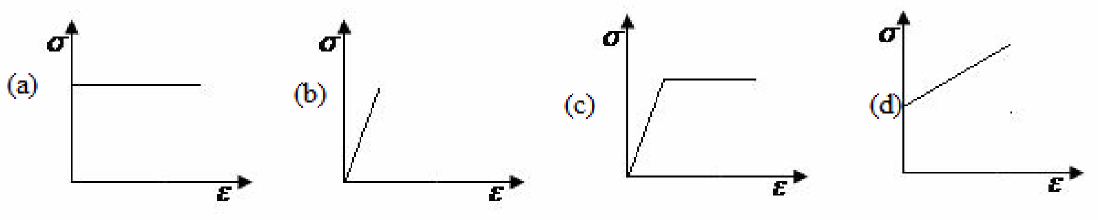
IAS-30. The stress-strain curve of an ideal elastic strain hardening material will be as



[IAS-1998]

IAS-30. Ans. (d)

IAS-31. An idealised stress-strain curve for a perfectly plastic material is given by



[IAS-1996]

IAS-31. Ans. (a)

IAS-32. Match List I with List II and select the correct answer using the codes given below the Lists: **[IAS-2002]**

List I **List II**

A. Ultimate strength **1.** Internal structure

B. Natural strain				**2.** Change of length per unit instantaneous length				
C. Conventional strain				**3.** Change of length per unit gauge length				
D. Stress				**4.** Load per unit area				

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	1	2	3	4	(b)	4	3	2	1
(c)	1	3	2	4	(d)	4	2	3	1

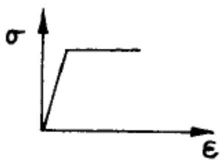
IAS-32. Ans. (a)

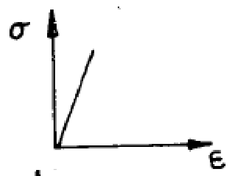
IAS-33. What is the cause of failure of a short MS strut under an axial load? [IAS-2007]
(a) Fracture stress (b) Shear stress (c) Buckling (d) Yielding

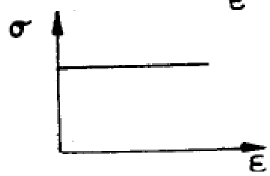
IAS-33. Ans. (d) In compression tests of ductile materials fractures is seldom obtained. Compression is accompanied by lateral expansion and a compressed cylinder ultimately assumes the shape of a flat disc.

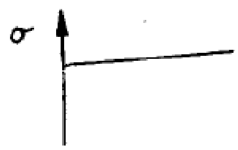
IAS-34. Match List I with List II and select the correct answer using the codes given the lists: [IAS-1995]

List I
A. Rigid-Perfectly plastic
B. Elastic-Perfectly plastic
C. Rigid-Strain hardening
D. Linearly elastic

List II
1.


2.


3.


4.


Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	3	1	4	2	(b)	1	3	2	4
(c)	3	1	2	4	(d)	1	3	4	2

IAS-34. Ans. (a)

IAS-35. Which one of the following materials is highly elastic? [IAS-1995]
(a) Rubber (b) Brass (c) Steel (d) Glass

IAS-35. Ans. (c) Steel is the highly elastic material because it is deformed least on loading, and regains its original from on removal of the load.

IAS-36. Assertion (A): Hooke's law is the constitutive law for a linear elastic material.
Reason (R) Formulation of the theory of elasticity requires the hypothesis that there exists a unique unstressed state of the body, to which the body returns whenever all the forces are removed. [IAS-2002]
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **not** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-36. Ans. (a)

IAS-37. Consider the following statements: [IAS-2002]

1. There are only two independent elastic constants.
2. Elastic constants are different in orthogonal directions.
3. Material properties are same everywhere.
4. Elastic constants are same in all loading directions.
5. The material has ability to withstand shock loading.

Which of the above statements are true for a linearly elastic, homogeneous and isotropic material?

(a) 1, 3, 4 and 5 (b) 2, 3 and 4 (c) 1, 3 and 4 (d) 2 and 5

IAS-37. Ans. (a)

IAS-38. Which one of the following pairs is NOT correctly matched? [IAS-1999]

(a) Uniformly distributed stress .... Force passed through the centroid of the cross-section
(b) Elastic deformation .... Work done by external forces during deformation is dissipated fully as heat
(c) Potential energy of strain .... Body is in a state of elastic deformation
(d) Hooke's law .... Relation between stress and strain

IAS-38. Ans. (b)

IAS-39. A tensile bar is stressed to 250 N/mm² which is beyond its elastic limit. At this stage the strain produced in the bar is observed to be 0.0014. If the modulus of elasticity of the material of the bar is 205000 N/mm² then the elastic component of the strain is very close to [IAS-1997]

(a) 0.0004 (b) 0.0002 (c) 0.0001 (d) 0.00005

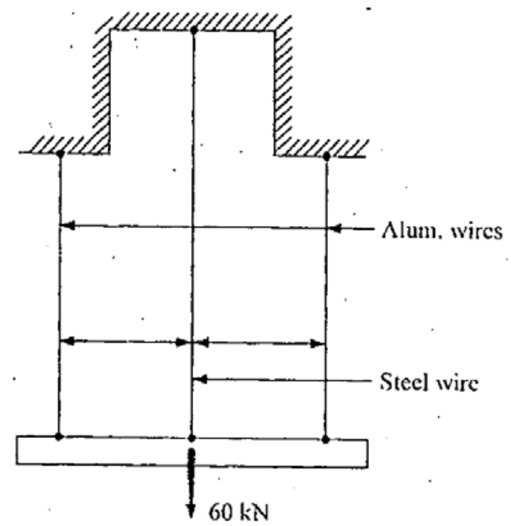
IAS-39. Ans. (b)

Previous Conventional Questions with Answers

Conventional Question IES-2010

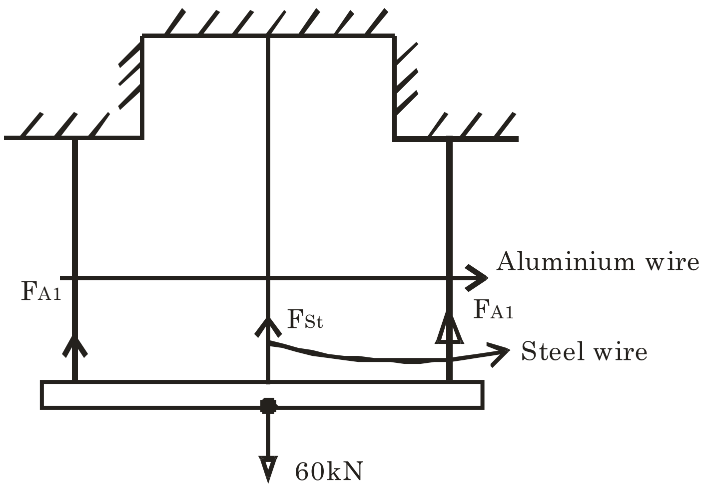
Q. **If a load of 60 kN is applied to a rigid bar suspended by 3 wires as shown in the above figure what force will be resisted by each wire?**

The outside wires are of Al, cross-sectional area 300 mm² and length 4 m. The central wire is steel with area 200 mm² and length 8 m.
Initially there is no slack in the wires $E = 2\times10^5 N/mm^2$ for Steel $= 0.667\times10^5 N/mm^2$ for Aluminum



[2 Marks]

Ans.



$$P = 60 \text{ kN}$$
$$a_{A1} = 300\,mm^2 \quad l_{A1} = 4\,m$$
$$a_{st} = 200\,mm^2 \quad l_{st} = 8\,m$$
$$E_{A1} = 0.667\times10^5 N/mm^2$$
$$E_{st} = 2\times10^5 N/mm^2$$

Force balance along vertical direction

$$2F_{A1} + F_{st} = 60 \text{ kN} \qquad (1)$$

Elongation will be same in all wires because rod is rigid remain horizontal after loading

$$\frac{F_{A1}\times l_{A1}}{a_{Al}.E_{Al}} = \frac{F_{st}.l_{st}}{a_{st}.E_{st}} \qquad (2)$$

$$\frac{F_{A1}\times 4}{300\times 0.667\times10^5} = \frac{F_{st}\times 8}{200\times 2\times10^5}$$

$$F_{A1} = 1.0005\ F_{st} \qquad (3)$$

From equation (1) $\mathbf{F_{st} = \frac{60 \times 10^3}{3.001} = 19.99\ kN\ or\ 20\ kN}$

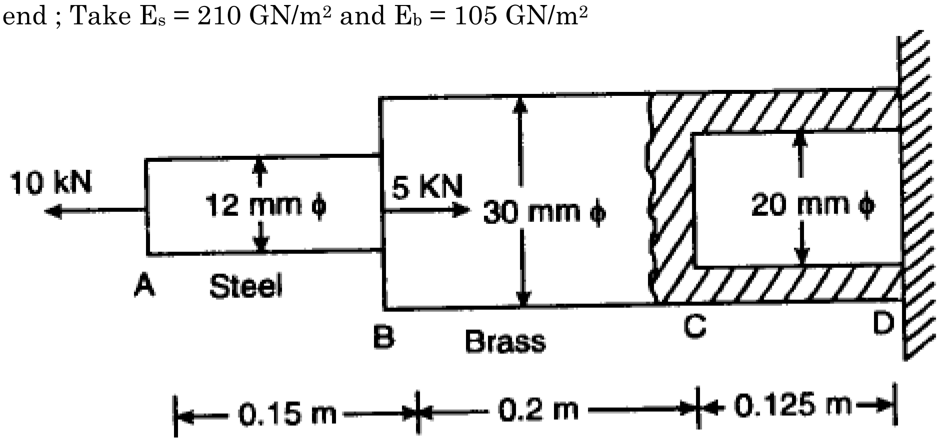
$\mathbf{F_{A1} = 20\ kN}$

$$\left.\begin{aligned}\mathbf{F_{A1}} &= \mathbf{20\ kN}\\ \mathbf{F_{st}} &= \mathbf{20\ kN}\end{aligned}\right\}\text{ Answer.}$$

Conventional Question GATE

***Question*:** **The diameters of the brass and steel segments of the axially loaded bar shown in figure are 30 mm and 12 mm respectively. The diameter of the hollow section of the brass segment is 20 mm.**

***Determine*:** (i) The maximum normal stress in the steel and brass (ii) The displacement of the free end ; Take $E_s = 210\ GN/m^2$ and $E_b = 105\ GN/m^2$



***Answer*:** $A_s = \frac{\pi}{4} \times (12)^2 = 36\pi\,mm^2 = 36\pi \times 10^{-6} m^2$

$$(A_b)_{BC} = \frac{\pi}{4} \times (30)^2 = 225\pi\,mm^2 = 225\pi \times 10^{-6} m^2$$

$$(A_b)_{CD} = \frac{\pi}{4} \times \left(30^2 - 20^2\right) = 125\pi\,mm^2 = 125\pi \times 10^{-6} m^2$$

(i) The maximum normal stress in steel and brass:

$$\sigma_s = \frac{10 \times 10^3}{36\pi \times 10^{-6}} \times 10^{-6} MN/m^2 = 88.42 MN/m^2$$

$$(\sigma_b)_{BC} = \frac{5 \times 10^3}{225\pi \times 10^{-6}} \times 10^{-6} MN/m^2 = 7.07 MN/m^2$$

$$(\sigma_b)_{CD} = \frac{5 \times 10^3}{125\pi \times 10^{-6}} \times 10^{-6} MN/m^2 = 12.73 MN/m^2$$

(ii) The displacement of the free end:

$$\delta l = (\delta l_s)_{AB} + (\delta l_b)_{BC} + (\delta l_b)_{CD}$$

$$= \frac{88.42 \times 0.15}{210 \times 10^9 \times 10^{-6}} + \frac{7.07 \times 0.2}{105 \times 10^9 \times 10^{-6}} + \frac{12.73 \times 0.125}{105 \times 10^9 \times 10^{-6}} \qquad \left(\because \delta l = \frac{\sigma l}{E}\right)$$

$$= 9.178 \times 10^{-5} m = 0.09178\ mm$$

Conventional Question IES-1999

***Question*:** **Distinguish between fatigue strength and fatigue limit.**

***Answer*:** Fatigue strength as the value of cyclic stress at which failure occurs after N cycles. And fatigue limit as the limiting value of stress at which failure occurs as N becomes very large (sometimes called infinite cycle)

Conventional Question IES-1999

Question: **List at least two factors that promote transition from ductile to brittle fracture.**

Answer: (i) With the grooved specimens only a small reduction in area took place, and the appearance of the facture was like that of brittle materials.

(ii) By internal cavities, thermal stresses and residual stresses may combine with the effect of the stress concentration at the cavity to produce a crack. The resulting fracture will have the characteristics of a brittle failure without appreciable plastic flow, although the material may prove ductile in the usual tensile tests.

Conventional Question IES-1999

Question: **Distinguish between creep and fatigue.**

Answer: Fatigue is a phenomenon associated with variable loading or more precisely to cyclic stressing or straining of a material, metallic, components subjected to variable loading get fatigue, which leads to their premature failure under specific conditions.

When a member is subjected to a constant load over a long period of time it undergoes a slow permanent deformation and this is termed as "Creep". This is dependent on temperature.

Conventional Question IES-2008

Question: **What different stresses set-up in a bolt due to initial tightening, while used as a fastener? Name all the stresses in detail.**

Answer: (i) When the nut is initially tightened there will be some elongation in the bolt so tensile stress will develop.

(ii) While it is tightening a torque across some shear stress. But when tightening will be completed there should be no shear stress.

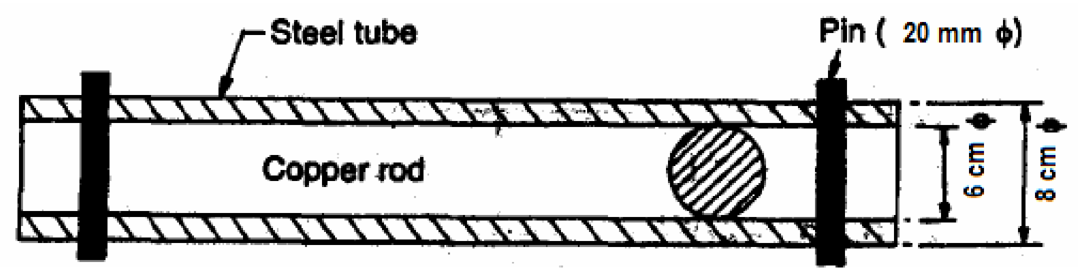
Conventional Question IES-2008

Question: **A Copper rod 6 cm in diameter is placed within a steel tube, 8 cm external diameter and 6 cm internal diameter, of exactly the same length. The two pieces are rigidly fixed together by two transverse pins 20 mm in diameter, one at each end passing through both rod and the tube.**

Calculated the stresses induced in the copper rod, steel tube and the pins if the temperature of the combination is raised by 50°C.

[Take Es=210 GPa, $\alpha_s = 0.0000115/^oC$; Ec=105 GPa, $\alpha_c = 0.000017/^oC$]

Answer:



$$\frac{\sigma_c}{E_c} + \frac{\sigma_s}{E_s} = \Delta t(\alpha_c - \alpha_s)$$

$$\text{Area of copper rod}(A_c) = \frac{\pi d^2}{4} = \frac{\pi}{4}\left(\frac{6}{100}\right)^2 m^2 = 2.8274\times10^{-3} m^2$$

$$\text{Area of steel tube }(A_s) = \frac{\pi d^2}{4} = \frac{\pi}{4}\left[\left(\frac{8}{100}\right)^2 - \left(\frac{6}{100}\right)^2\right] m^2 = 2.1991\times10^{-3} m^2$$

Rise in temperature, $\Delta t = 50^oC$

Free expansion of copper bar $=\alpha_c L \Delta t$

Free expansion of steel tube $=\alpha_s L \Delta t$

Difference in free expansion $=(\alpha_c - \alpha_s)L\triangle t$

$=(17\text{-}11.5)\times10^{-6}\times L\times 50=2.75\times10^{-4}L\,m$

A compressive force (P) exerted by the steel tube on the copper rod opposed the extra expansion of the copper rod and the copper rod exerts an equal tensile force P to pull the steel tube. In this combined effect reduction in copper rod and increase in length of steel tube equalize the difference in free expansions of the combined system.

Reduction in the length of copper rod due to force P Newton=

$$(\triangle L)_C = \frac{PL}{A_c E_c} = \frac{PL}{(2.8275\times10^{-3})(105\times10^{9})}\text{m}$$

Increase in length of steel tube due to force P

$$(\triangle L)_S = \frac{PL}{A_s E_s} = \frac{P.L}{(2.1991\times10^{-3})(210\times10^{9})}m$$

Difference in length is equated

$$(\triangle L)_c + (\triangle L)_s = 2.75\times10^{-4}L$$

$$\frac{PL}{(2.8275\times10^{-3})(105\times10^{9})} + \frac{P.L}{(2.1991\times10^{-3})(210\times10^{9})} = 2.75\times10^{-4}L$$

Or P = 49.695 kN

$$\text{Stress in copper rod, } \sigma_c = \frac{P}{A_c} = \frac{49695}{2.8275\times10^{-3}}\text{MPa}=17.58\text{MPa}$$

$$\textit{Stress}\text{ in steel tube, } \sigma_s = \frac{P}{A_s} = \frac{49695}{2.1991\times10^{-3}}\text{MPa} = 22.6\text{MPa}$$

Since each of the pin is in double shear, shear stress in pins (τ_{pin})

$$= \frac{P}{2\times A_{pin}} = \frac{49695}{2\times\frac{\pi}{4}(0.02)^2} = 79\text{ MPa}$$

Conventional Question IES-2002

***Question*:** **Why are the bolts, subjected to impact, made longer?**

***Answer*:** If we increase length its volume will increase so shock absorbing capacity will increased.

Conventional Question IES-2007

***Question*:** **Explain the following in brief:**

(i) Effect of size on the tensile strength

(ii) Effect of surface finish on endurance limit.

***Answer*:** (i) When size of the specimen increases tensile strength decrease. It is due to the reason that if size increases there should be more change of defects (voids) into the material which reduces the strength appreciably.

(ii) If the surface finish is poor, the endurance strength is reduced because of scratches present in the specimen. From the scratch crack propagation will start.

Conventional Question IES-2004

***Question*:** **Mention the relationship between three elastic constants i.e. elastic modulus (E), rigidity modulus (G), and bulk modulus (K) for any Elastic material. How is the Poisson's ratio (μ) related to these modulli?**

Answer: $E = \dfrac{9KG}{3K+G}$

$$E = 3K(1-2\mu) = 2G(1+\mu) = \frac{9KG}{3K+G}$$

Conventional Question IES-1996

***Question*:** **The elastic and shear moduli of an elastic material are 2×10^{11} Pa and 8×10^{10} Pa respectively. Determine Poisson's ratio of the material.**

***Answer*:** We know that $E = 2G(1+\mu) = 3K(1-2\mu) = \dfrac{9KG}{3K+G}$

$$\text{or,}\ 1+\mu = \frac{E}{2G}$$

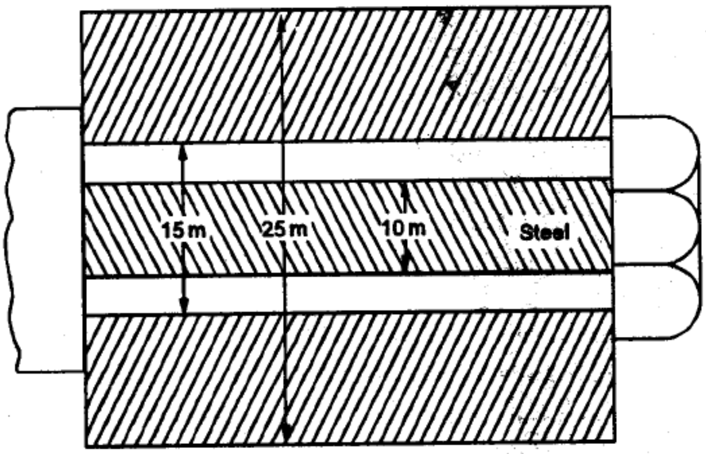
$$\text{or}\ \mu = \frac{E}{2G} - 1 = \frac{2\times10^{11}}{2\times(8\times10^{10})} - 1 = 0.25$$

Conventional Question IES-2003

***Question*:** **A steel bolt of diameter 10 mm passes through a brass tube of internal diameter 15 mm and external diameter 25 mm. The bolt is tightened by a nut so that the length of tube is reduced by 1.5 mm. If the temperature of the assembly is raised by 40°C, estimate the axial stresses the bolt and the tube before and after heating. Material properties for steel and brass are:**

$E_S = 2\times10^5\ \mathbf{N/mm^2}$ $\alpha_S = 1.2\times10^{-5}/^oC$ and $E_b = \mathbf{1\times10^5\ N/mm^2}$ $\alpha_b = 1.9\times10^{-15}/^oC$

***Answer*:**



$$\textit{Area}\ \text{of steel bolt}\ (A_s) = \frac{\pi}{4}\times(0.010)^2 m^2 = 7.854\times10^{-5} m^2$$

$$\textit{Area}\ \text{of brass tube}\ (A_b) = \frac{\pi}{4}\left[(0.025)^2 - (0.015)^2\right] = 3.1416\times10^{-4}$$

<u>Stress due to tightening of the nut</u>

Compressive force on brass tube= tensile fore on steel bolt

$$\text{or,}\ \sigma_b A_b = \sigma_S A_s$$

$$\textit{or},\ E_{b.}\frac{(\Delta l)_b}{\ell}.A_b = \sigma_s A_s \qquad \left[\because E = \frac{\sigma}{\in} = \frac{\sigma}{\left(\frac{\Delta L}{L}\right)}\right]$$

Let assume total length (ℓ)=1m

$$\text{Therefore}\ \ (1\times10^5\times10^6)\times\frac{(1.5\times10^{-3})}{1}\times\left(3.1416\times10^{-4}\right) = \sigma_s\times7.854\times10^{-5}$$

$$\textit{or}\quad \sigma_s = 600\,MPa\ (\textit{tensile})$$

$$\text{and}\ \sigma_b = E_{b.}\frac{(\Delta l)_b}{\ell} = (1\times10^5)\times\frac{(1.5\times10^{-3})}{1}MPa = 150MPa\,(\textit{Compressive})$$

So before heating

Stress in brass tube $(\sigma_b) = 150 MPa (compressive)$

Stress in steel bolt$(\sigma_s) = 600 MPa$ (tensile)

Stress due to rise of temperature

Let stress $\sigma_b^{'}$ & $\sigma_s^{'}$ *are* due to brass tube and steel bolt.

If the two members had been free to expand,

Free expansion of steel = $\alpha_s \times \Delta t \times 1$

Free expansion of brass tube = $\alpha_b \times \Delta t \times 1$

Since $\alpha_b > \sigma_s$ free expansion of copper is greater than the free expansion of steel. But they are rigidly fixed so final expansion of each members will be same. Let us assume this final expansion is 'δ', The free expansion of brass tube is grater than δ, while the free expansion of steel is less than δ. Hence the steel rod will be subjected to a tensile stress while the brass tube will be subjected to a compressive stress.

For the equilibrium of the whole system,

Total tension (Pull) in steel =Total compression (Push) in brass tube.

$$\sigma_b^{'} A_b = \sigma_s^{'} A_s \; or, \;\; \sigma_b^{'} = \sigma_s^{'} \times \frac{A_s}{A_b} = \frac{7.854 \times 10^{-5}}{3.14 \times 10^{-4}} \sigma_S^{'} = 0.25 \sigma_S^{'}$$

Final expansion of steel =final expansion of brass tube

$$\alpha_s (\Delta t).1 + \frac{\sigma_s^{'}}{E_s} \times 1 = \alpha_b (\Delta t) \times 1 - \frac{\sigma_b^{'}}{E_b} \times 1$$

$$or, \left(1.2 \times 10^{-5}\right) \times 40 \times 1 + \frac{\sigma_s^{'}}{2 \times 10^5 \times 10^6} = (1.9 \times 10^{-5}) \times 40 \times 1 - \frac{\sigma_b^{'}}{1 \times 10^5 \times 10^6} - -(ii)$$

From(i) & (ii) we get

$$\sigma_s^{'} \left[\frac{1}{2 \times 10^{11}} + \frac{0.25}{10^{11}} \right] = 2.8 \times 10^{-4}$$

$or, \sigma_s^{'} = 37.33$ MPa (Tensile stress)

or, $\sigma_b^{'}$= 9.33MPa (compressive)

Therefore, the final stresses due to tightening and temperature rise

Stress in brass tube =$\sigma_b + \sigma_b^{'}$=150+9.33MPa=159.33MPa

Stress in steel bolt =$\sigma_s + \sigma_s^{'}$= 600 + 37.33 = 637.33MPa.

Conventional Question IES-1997

***Question*:** **A Solid right cone of axial length h is made of a material having density ρ and elasticity modulus E. It is suspended from its circular base. Determine its elongation due to its self weight.**

***Answer*:** See in the figure MNH is a solid right cone of length 'h' .

Let us assume its wider end of diameter'd' fixed rigidly at MN.

Now consider a small strip of thickness dy at a distance y from the lower end.

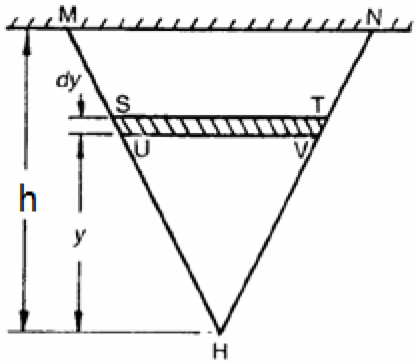
Let 'ds' is the diameter of the strip.

$$\therefore \text{Weight of portion UVH} = \frac{1}{3}\left(\frac{\pi d_s^2}{4}\right) y \times \rho g - (i)$$

From the similar triangles MNH and UVH,

$$\frac{MN}{UV}=\frac{d}{d_s}=\frac{\ell}{y}$$

$$or, d_s=\frac{d.y}{\ell}----(ii)$$



$$\therefore \text{Stress at section UV} = \frac{\text{force at UV}}{cross-section \text{ area at UV}}=\frac{\text{Weight of UVH}}{\left(\frac{\pi d_s^2}{4}\right)}$$

$$=\frac{\frac{1}{3}.\frac{\pi d_s^2}{4}.y.\rho g}{\left(\frac{\pi d_s^2}{4}\right)}=\frac{1}{3}y\rho g$$

$$So, \text{ extension in dy}=\frac{\left(\frac{1}{3}y\rho g\right).dy}{E}$$

$$\therefore \text{Total extension of the bar} = \int_0^h \frac{\frac{1}{3}y\rho g dy}{E}=\frac{\rho g h^2}{6E}$$

From stress-strain relation ship

$$E=\frac{\delta}{\in}=\frac{\delta}{\frac{d\ell}{\ell}} or, d\ell=\frac{\delta.\ell}{E}$$

Conventional Question IES-2004

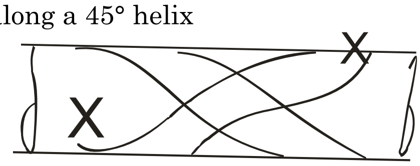
Question: **Which one of the three shafts listed hare has the highest ultimate tensile strength? Which is the approximate carbon content in each steel?**
(i) Mild Steel (ii) cast iron (iii) spring steel

Answer: Among three steel given, spring steel has the highest ultimate tensile strength.
Approximate carbon content in
(i) Mild steel is (0.3% to 0.8%)
(ii) Cost iron (2% to 4%)
(iii) Spring steel (0.4% to 1.1%)

Conventional Question IES-2003

Question: **If a rod of brittle material is subjected to pure torsion, show with help of a sketch, the plane along which it will fail and state the reason for its failure.**

Answer: Brittle materials fail in tension. In a torsion test the maximum tensile test Occurs at 45° to the axis of the shaft. So failure will occurs along a 45° to the axis of the shaft. So failure will occurs along a 45° helix

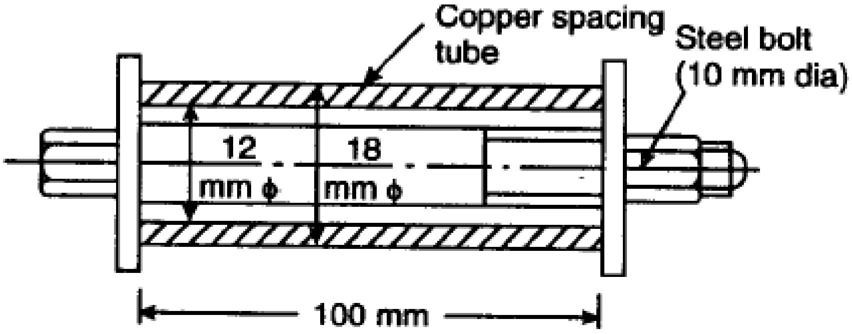


So failures will occurs according to 45° plane.

Conventional Question IAS-1995

Question: **The steel bolt shown in Figure has a thread pitch of 1.6 mm. If the nut is initially tightened up by hand so as to cause no stress in the copper spacing tube, calculate the stresses induced in the tube and in the bolt if a spanner is then used to turn the nut through 90°.Take E_c and E_s as 100 GPa and 209 GPa respectively.**

Answer: Given: p = 1.6 mm, E_c= 100 GPa ; E_s = 209 CPa.



Stresses induced in the tube and the bolt, σ_c, σ_s :

$$A_s = \frac{\pi}{4} \times \left(\frac{10}{1000}\right)^2 = 7.584 \times 10^{-5} m^2$$

$$A_s = \frac{\pi}{4} \times \left[\left(\frac{18}{1000}\right)^2 - \left(\frac{12}{1000}\right)^2\right] = 14.14 \times 10^{-5} m^2$$

Tensile force on steel bolt, P_s = compressive force in copper tube, P_c = P

Also, Increase in length of bolt + decrease in length of tube = axial displacement of nut

$$\text{i,e} \quad (\delta l)_s + (\delta l)_c = 1.6 \times \frac{90}{360} = 0.4 mm = 0.4 \times 10^{-3} m$$

$$\text{or} \quad \frac{Pl}{A_s E_s} + \frac{Pl}{A_c E_c} = 0.4 \times 10^{-3} \qquad (\because l_s = l_c = l)$$

$$\text{or} \quad P \times \left(\frac{100}{1000}\right)\left[\frac{1}{7.854 \times 10^{-5} \times 209 \times 10^9} + \frac{1}{14.14 \times 10^{-5} \times 100 \times 10^9}\right] = 0.4 \times 10^{-3}$$

$$\text{or} \quad P = 30386 N$$

$$\therefore \frac{P}{A_s} = 386.88 MPa \qquad \text{and } \frac{P}{A_c} = 214.89 MPa$$

Conventional Question AMIE-1997

Question: **A steel wire 2 m long and 3 mm in diameter is extended by 0·75 mm when a weight W is suspended from the wire. If the same weight is suspended from a brass wire, 2·5 m long and 2 mm in diameter, it is elongated by 4 -64 mm. Determine the modulus of elasticity of brass if that of steel be 2.0 × 10^5 N / mm^2**

Answer: Given, l_s = 2 m, d_s = 3 mm, δl_s = 0·75 mm; E_s = 2·0 × 10^5 N / mm^2; l_b = 2.5 m, d_b =2 mm δl_b = 4.64m m and let modulus of elasticity of brass = E_b

Hooke's law gives, $\delta l = \dfrac{Pl}{AE}$ [Symbol has usual meaning]

Case I: For steel wire:

$$\delta l_s = \frac{Pl_s}{A_sE_s}$$

$$\text{or } 0.75 = \frac{P \times (2 \times 1000)}{\left(\frac{\pi}{4} \times 3^2\right) \times 2.0 \times 10^5 \times \frac{1}{2000}} \qquad \text{---- (i)}$$

Case II: For bass wire:

$$\delta l_b = \frac{Pl_b}{A_bE_b}$$

$$4.64 = \frac{P \times (2.5 \times 1000)}{\left(\frac{\pi}{4} \times 2^2\right) \times E_b} \qquad \text{---- (ii)}$$

$$\text{or} \qquad P = 4.64 \times \left(\frac{\pi}{4} \times 2^2\right) \times E_b \times \frac{1}{2500}$$

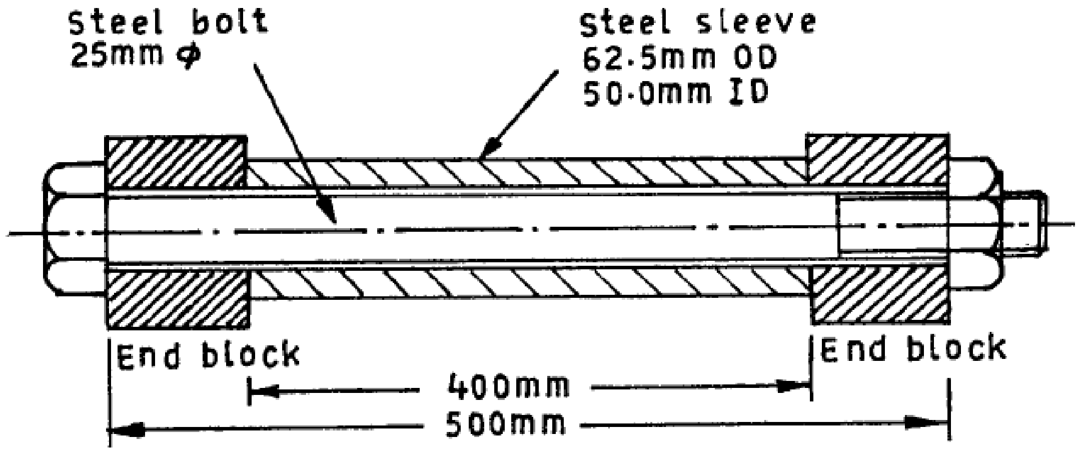
From (i) and (ii), we get

$$0.75 \times \left(\frac{\pi}{4} \times 3^2\right) \times 2.0 \times 10^5 \times \frac{1}{2000} = 4.64 \times \left(\frac{\pi}{4} \times 2^2\right) \times E_b \times \frac{1}{2500}$$

$$\text{or} \qquad E_b = 0.909 \times 10^5 N/mm^2$$

Conventional Question AMIE-1997

Question: **A steel bolt and sleeve assembly is shown in figure below. The nut is tightened up on the tube through the rigid end blocks until the tensile force in the bolt is 40 kN. If an external load 30 kN is then applied to the end blocks, tending to pull them apart, estimate the resulting force in the bolt and sleeve.**



Answer: Area of steel bolt, $A_b = \left(\frac{25}{1000}\right)^2 = 4.908 \times 10^{-4} m^2$

Area of steel sleeve, $A_s = \frac{\pi}{4}\left[\left(\frac{62.5}{1000}\right)^2 - \left(\frac{50}{1000}\right)^2\right] = 1.104 \times 10^{-3} m^2$

Forces in the bolt and sleeve:

(i) Stresses due to tightening the nut:

Let σ_b = stress developed in steel bolt due to tightening the nut; and

σ_s = stress developed in steel sleeve due to tightening the nut.

Tensile force in the steel bolt = 40 kN = 0·04 MN

$$\sigma_b \times A_b = 0.04$$

or $\sigma_b \times 4.908 \times 10^{-4} = 0.04$

$$\therefore \quad \sigma_b = \frac{0.04}{4.908 \times 10^{-4}} = 81.5 MN/m^2 \left(\text{tensile}\right)$$

Compressive force in steel sleeve = 0·04 MN

$$\sigma_s \times A_s = 0.04$$

or $\sigma_s \times 1.104 \times 10^{-3} = 0.04$

$$\therefore \quad \sigma_s = \frac{0.04}{1.104 \times 10^{-3}} = 36.23 MN/m^2 \left(\text{compressive}\right)$$

(ii) Stresses due to tensile force:

Let the stresses developed due to tensile force of 30 kN = 0·03 MN in steel bolt and sleeve be σ'_b and σ'_s respectively.

Then, $\sigma'_b \times A_b + \sigma'_s \times A_s = 0.03$

$$\sigma'_b \times 4.908 \times 10^{-4} + \sigma'_s \times 1.104 \times 10^{-3} = 0.03 \qquad ---(i)$$

In a compound system with an external tensile load, elongation caused in each will be the same.

$$\delta l_b = \frac{\sigma'_b}{E_b} \times l_b$$

or $\delta l_b = \frac{\sigma'_b}{E_b} \times 0.5 \qquad \left(\text{Given}, l_b = 500 mm = 0.5\right)$

and $\delta l_s = \frac{\sigma'_s}{E_s} \times 0.4 \qquad \left(\text{Given}, l_s = 400 mm = 0.4\right)$

But $\delta l_b = \delta_s$

$$\therefore \quad \frac{\sigma'_b}{E_b} \times 0.5 = \frac{\sigma'_s}{E_s} \times 0.4$$

or $\sigma'_b = 0.8 \sigma'_s \qquad \left(\text{Given}, E_b = E_s\right) \qquad ---(2)$

Substituting this value in (1), we get

$$0.8\sigma'_s \times 4.908 \times 10^{-4} + \sigma'_s \times 1.104 \times 10^{-3} = 0.03$$

gives $\sigma'_s = 20 MN/m^2 \left(\text{tensile}\right)$

and $\sigma'_b = 0.8 \times 20 = 16 MN/m^2 \left(\text{tensile}\right)$

Resulting stress in steel bolt,

$$\left(\sigma_b\right)_r = \sigma_b + \sigma'_b = 81.5 + 16 = 97.5 MN/m^2$$

Resulting stress in steel sleeve,

$$\left(\sigma_s\right)_r = \sigma_s + \sigma'_s = 36.23 - 20 = 16.23 MN/m^2 \left(\text{compressive}\right)$$

Resulting force in steel bolt, $= \left(\sigma_b\right)_r \times A_b$

$$= 97.5 \times 4.908 \times 10^{-4} = 0.0478 MN \left(\text{tensile}\right)$$

Resulting force in steel sleeve $= \left(\sigma_b\right)_r \times A_s$

$$= 16.23 \times 1.104 \times 10^{-3} = 0.0179 MN \left(\text{compressive}\right)$$

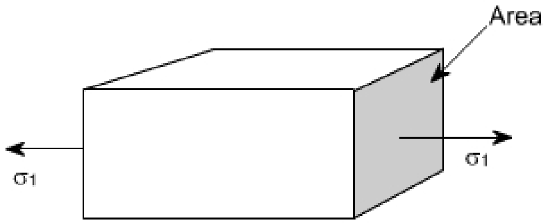
2. Principal Stress and Strain

Theory at a Glance (for IES, GATE, PSU)

2.1 States of stress

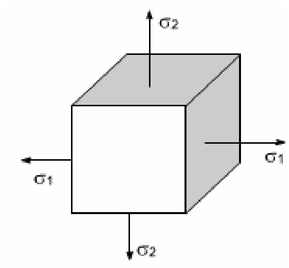
- ***Uni-axial stress:*** only one non-zero principal stress, i.e. σ_1

Right side figure represents Uni-axial state of stress.



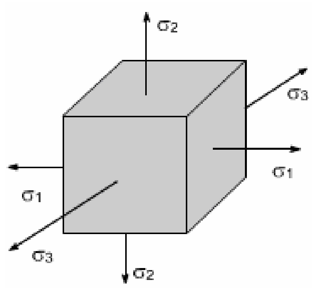
- ***Bi-axial stress:*** one principal stress equals zero, two do not, i.e. $\sigma_1 > \sigma_3$; $\sigma_2 = 0$

Right side figure represents Bi-axial state of stress.



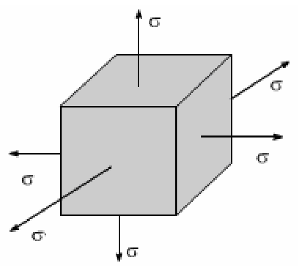
- ***Tri-axial stress:*** three non-zero principal stresses, i.e. $\sigma_1 > \sigma_2 > \sigma_3$

Right side figure represents Tri-axial state of stress.



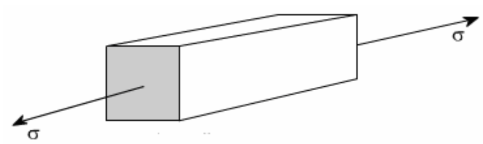
- ***Isotropic stress:*** three principal stresses are equal, i.e. $\sigma_1 = \sigma_2 = \sigma_3$

Right side figure represents isotropic state of stress.



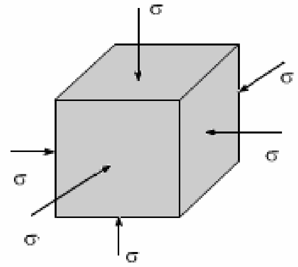
- ***Axial stress:*** two of three principal stresses are equal, i.e. $\sigma_1 = \sigma_2$ or $\sigma_2 = \sigma_3$

Right side figure represents axial state of stress.

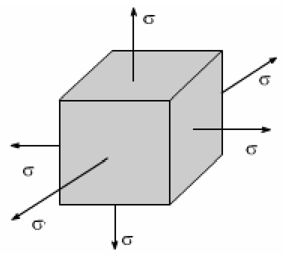


- ***Hydrostatic pressure:*** weight of column of fluid in interconnected pore spaces.
  $P_{hydrostatic} = \rho_{fluid}\, gh$ (density, gravity, depth)
- ***Hydrostatic stress:*** Hydrostatic stress is used to describe a state of tensile or compressive stress equal in all directions within or external to a body. Hydrostatic stress causes a change in volume of a material. Shape of the body remains unchanged i.e. no distortion occurs in the body.

Right side figure represents Hydrostatic state of stress.



Or



2.2 Uni-axial stress on oblique plane

Let us consider a bar of uniform cross sectional area A under direct tensile load P giving rise to axial normal stress P/A acting on a cross section XX. Now consider another section given by the plane YY inclined at θ with the XX. This is depicted in following three ways.

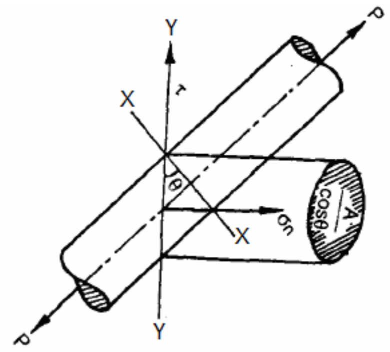


Fig. (a)

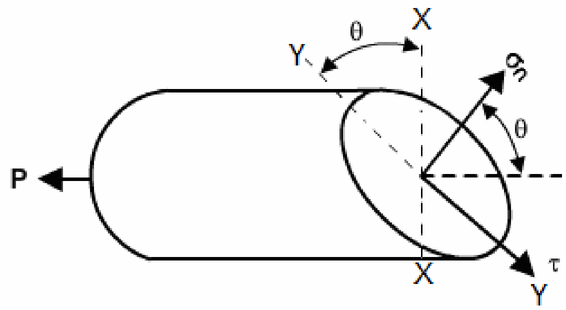


Fig. (b)

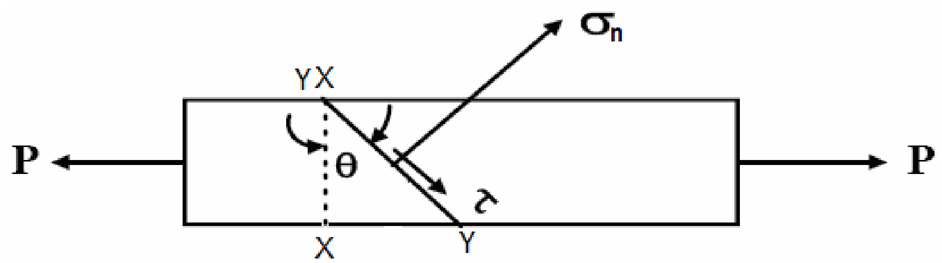


Fig. (c)

Area of the YY Plane $= \dfrac{A}{\cos\theta}$; Let us assume the ***normal stress*** in the YY plane is σ_n and there is a ***shear stress*** τ acting parallel to the YY plane.

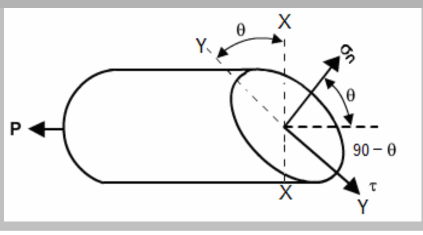
Now resolve the force P in two perpendicular direction one normal to the plane YY = $P\cos\theta$ and another parallel to the plane YY = Pcosθ

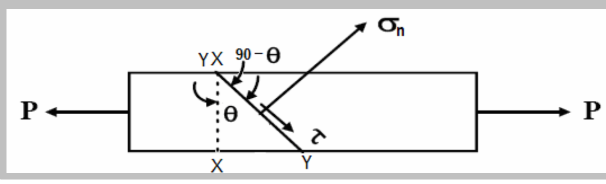
Therefore equilibrium gives, $\sigma_n \dfrac{A}{\cos\theta} = P\cos\theta$ **or** $$\boxed{\sigma_n = \frac{P}{A}\cos^2\theta}$$

and $\tau \times \dfrac{A}{\cos\theta} = P\sin\theta$ **or** $\tau = \dfrac{P}{A}\sin\theta\cos\theta$ **or** $$\boxed{\tau = \frac{P}{2A}\sin 2\theta}$$

- Note the variation of ***normal stress*** σ_n and ***shear stress*** τ with the variation of θ. When $\theta = 0$, normal stress σ_n is maximum i.e. $(\sigma_n)_{\max} = \dfrac{P}{A}$ and shear stress $\tau = 0$. As θ is increased, the normal stress σ_n diminishes, until when $\theta = 0,\ \sigma_n = 0$. But if angle θ increased shear stress τ increases to a maximum value $\tau_{\max} = \dfrac{P}{2A}$ at $\theta = \dfrac{\pi}{4} = 45^o$ and then diminishes to $\tau = 0$ at $\theta = 90^o$
- The shear stress will be maximum when $\sin 2\theta = 1 \ or\ \theta = 45^o$
- And the maximum shear stress, $\tau_{\max} = \dfrac{P}{2A}$
- In ductile material failure in tension is initiated by shear stress i.e. the failure occurs across the shear planes at 45^o (where it is maximum) to the applied load.

Let us clear a concept about a common mistake: The angle θ is not between the applied load and the plane. It is between the planes XX and YY. But if in any question the angle between the applied load and the plane is given don't take it as θ. The angle between the applied load and the plane is 90 - θ. In this case you have to use the above formula as $\sigma_n = \dfrac{P}{A}\cos^2(90-\theta)$ and $\tau = \dfrac{P}{2A}\sin(180-2\theta)$ where θ is the angle between the applied load and the plane. Carefully observe the following two figures it will be clear.





Let us take an example: A metal block of 100 mm^2 cross sectional area carries an axial tensile load of 10 kN. For a plane inclined at 30^0 with the direction of applied load, calculate:

(a) Normal stress

(b) Shear stress

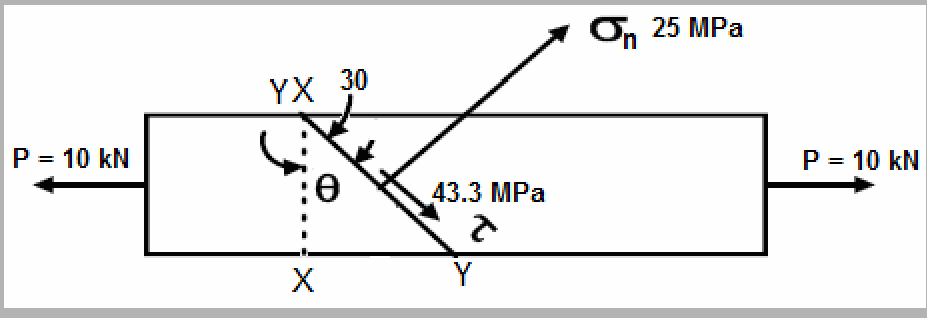
(c) Maximum shear stress.

Answer: Here $\theta = 90^o - 30^o = 60^o$

(a) Normal stress $(\sigma_n) = \frac{P}{A}\cos^2\theta = \frac{10\times10^3 N}{100\,mm^2}\times\cos^2 60^o = 25MPa$

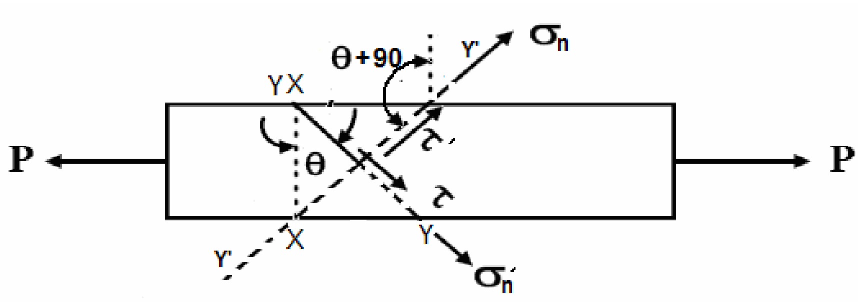
(b) Shear stress $(\tau) = \frac{P}{2A}\sin 2\theta = \frac{10\times10^3 N}{2\times100\,mm^2}\times\sin 120^o = 43.3MPa$

(c) Maximum shear stress $(\tau_{max}) = \frac{P}{2A} = \frac{10\times10^3 N}{2\times100\,mm^2} = 50MPa$



- **Complementary stresses**

Now if we consider the stresses on an oblique plane Y'Y' which is perpendicular to the previous plane YY. The stresses on this plane are known as complementary stresses. ***Complementary normal stress is*** σ_n' and ***complementary shear stress is*** τ'. The following figure shows all the four stresses. To obtain the stresses σ_n' and τ' we need only to replace θ by $\theta+90^0$ in the previous equation. The angle $\theta+90^0$ is known as ***aspect angle.***



Therefore

$$\sigma_n' = \frac{P}{A}\cos^2\left(90^o+\theta\right) = \frac{P}{A}\sin^2\theta$$

$$\tau' = \frac{P}{2A}\sin 2\left(90^o+\theta\right) = -\frac{P}{2A}\sin 2\theta$$

It is clear $\sigma_n' + \sigma_n = \frac{P}{A}$ and $\tau' = -\tau$

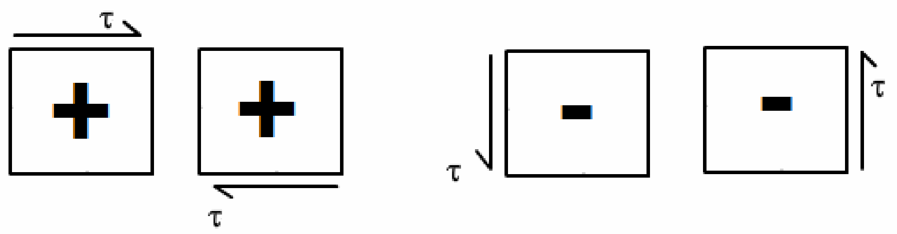
i.e. Complementary shear stresses are always equal in magnitude but opposite in sign.

- **Sign of Shear stress**

For sign of shear stress following rule have to be followed:

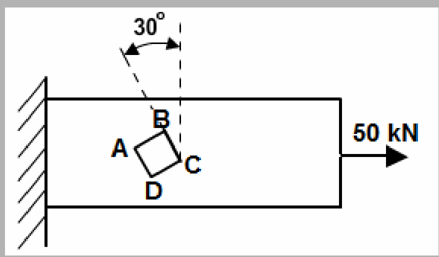
The shear stress τ on any face of the element will be considered ***positive*** when it has a ***clockwise*** moment with respect to a centre inside the element. If the moment is ***counter-clockwise*** with respect to a centre inside the element, the shear stress in ***negative.***

Note: The convention is opposite to that of moment of force. Shear stress tending to turn clockwise is positive and tending to turn counter clockwise is negative.



Let us take an example: A prismatic bar of 500 mm² cross sectional area is axially loaded with a tensile force of 50 kN. Determine all the stresses acting on an element which makes 30^0 inclination with the vertical plane.

Answer: Take an small element ABCD in 30^0 plane as shown in figure below,
Given, Area of cross-section, A = 500 mm², Tensile force (P) = 50 kN



Normal stress on 30° inclined plane, $(\sigma_n) = \frac{P}{A}\cos^2\theta = \frac{50\times10^3\,N}{500\ mm^2}\times\cos^2 30^o = 75MPa$ (+ive means tensile).

Shear stress on 30° planes, $(\tau) = \frac{P}{2A}\sin 2\theta = \frac{50\times10^3\,N}{2\times500\ mm^2}\times\sin(2\times30^o) = 43.3MPa$

(+ive means clockwise)

Complementary stress on $(\theta) = 90 + 30 = 120^o$

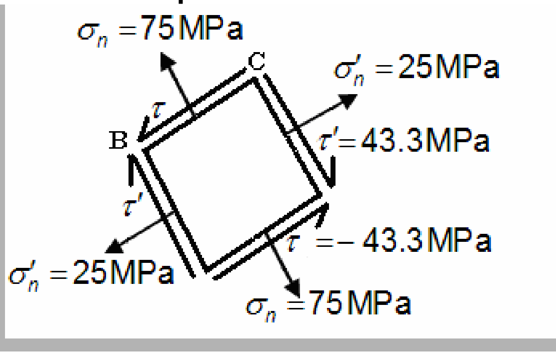
Normal stress on 120^0 inclined plane, $(\sigma_n') = \frac{P}{A}\cos^2\theta = \frac{50\times10^3\,N}{500\,mm^2}\times\cos^2 120^o = 25MPa$

(+ ive means tensile)

Shear stress on 120^0 nclined plane, $(\tau') = \frac{P}{2A}\sin 2\theta = \frac{50\times10^3\,N}{2\times500\,mm^2}\times\sin(2\times120^o) = -43.3MPa$

(- ive means counter clockwise)

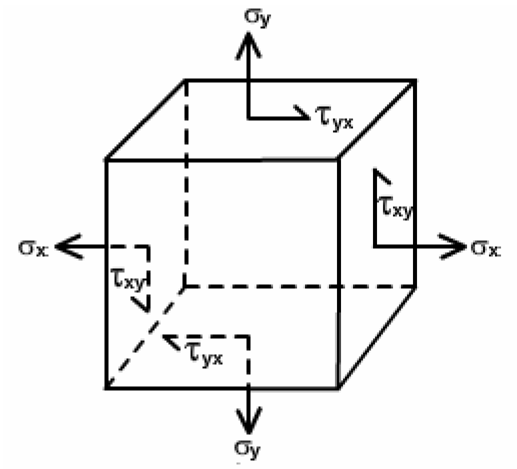
State of stress on the element ABCD is given below (magnifying)



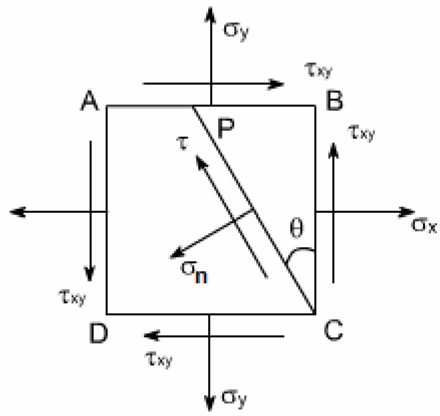
2.3 Complex Stresses (2-D Stress system)

i.e. Material subjected to combined direct and shear stress

We now consider a complex stress system below. The given figure ABCD shows on small element of material



Stresses in three dimensional element



Stresses in cross-section of the element

σ_x and σ_y are normal stresses and may be tensile or compressive. We know that normal stress may come from direct force or bending moment. τ_{xy} is shear stress. We know that shear stress may comes from direct shear force or torsion and τ_{xy} and τ_{yx} are complementary and

$$\tau_{xy} = \tau_{yx}$$

Let σ_n is the normal stress and τ is the shear stress on a plane at angle θ.

Considering the equilibrium of the element we can easily get

$$\text{Normal stress}\left(\sigma_n\right) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

and

$$\text{Shear stress}\left(\tau\right) = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

Above two equations are coming from considering equilibrium. They do not depend on material properties and are valid for elastic and in elastic behavior.

• Location of planes of maximum stress

(a) Normal stress, $(\sigma_n)_{max}$

For σ_n maximum or minimum

$$\frac{\partial \sigma_n}{\partial \theta} = 0, \text{ where } \sigma_n = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$or\ -\frac{(\sigma_x - \sigma_y)}{2} \times (\sin 2\theta) \times 2 + \tau_{xy}(\cos 2\theta) \times 2 = 0 \quad or\ \tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

(b) Shear stress, τ_{max}

For τ maximum or minimum

$$\frac{\partial \tau}{\partial \theta} = 0, \text{ where } \tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$or\ \frac{\sigma_x - \sigma_y}{2}(\cos 2\theta) \times 2 - \tau_{xy}(-\sin 2\theta) \times 2 = 0$$

$$or\ \cot 2\theta = \frac{-\tau_{xy}}{\sigma_x - \sigma_y}$$

Let us take an example: At a point in a crank shaft the stresses on two mutually perpendicular planes are 30 MPa (tensile) and 15 MPa (tensile). The shear stress across these planes is 10 MPa. Find the normal and shear stress on a plane making an angle 30^0 with the plane of first stress. Find also magnitude and direction of resultant stress on the plane.

Answer: Given $\sigma_x = +25\text{MPa}(\text{tensile}), \sigma_y = +15\text{MPa}(\text{tensile}), \tau_{xy} = 10\text{MPa and } 40^0$

$$\text{Therefore, Normal stress}(\sigma_n) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

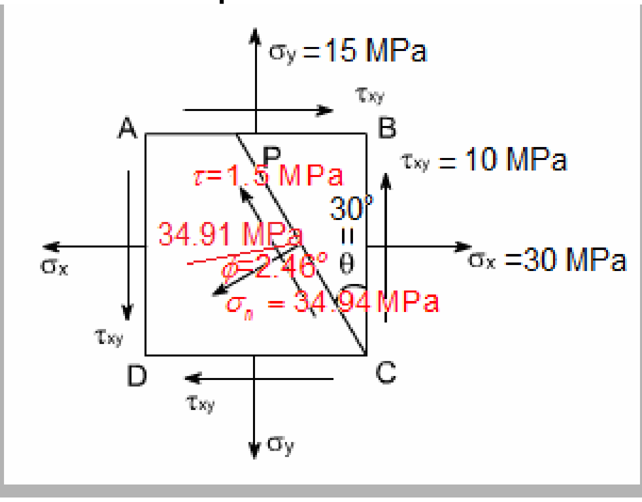
$$= \frac{30+15}{2} + \frac{30-15}{2}\cos(2\times 30^o) + 10\sin(2\times 30^o) = 34.91\text{ MPa}$$

$$\text{Shear stress }(\tau) = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{30-15}{2}\sin(2\times 30^o) - 10\cos(2\times 30^o) = 1.5\text{MPa}$$

$$\text{Resultant stress }(\sigma_r) = \sqrt{(34.91)^2 + 1.5^2} \quad = 34.94\text{MPa}$$

$$\text{and Obliquity}(\phi), \tan\phi = \frac{\tau}{\sigma_n} = \frac{1.5}{34.91} \quad \Rightarrow \phi = 2.46^0$$

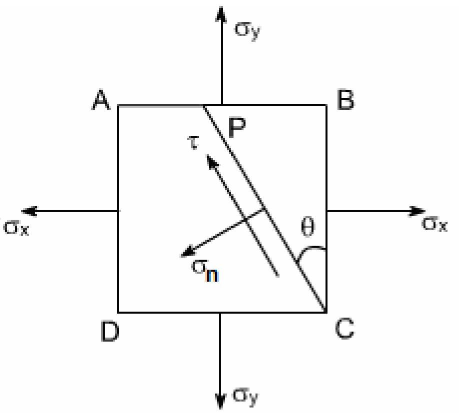


2.4 Bi-axial stress

Let us now consider a stressed element ABCD where τ_{xy} =0, i.e. only σ_x and σ_y is there. This type of stress is known as bi-axial stress. In the previous equation if you put τ_{xy} =0 we get Normal stress, σ_n and shear stress, τ on a plane at angle θ.

- Normal stress , $\sigma_n = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2} cos 2\theta$
- Shear/Tangential stress, $\tau = \dfrac{\sigma_x - \sigma_y}{2} \sin 2\theta$
- For complementary stress, aspect angle = $\theta + 90^0$
- Aspect angle 'θ' varies from 0 to $\pi/2$
- Normal stress σ_n varies between the values

  $\sigma_x (\theta = 0)$ & $\sigma_y (\theta = \pi/2)$



Let us take an example: The principal tensile stresses at a point across two perpendicular planes are 100 MPa and 50 MPa. Find the normal and tangential stresses and the resultant stress and its obliquity on a plane at 20⁰ with the major principal plane

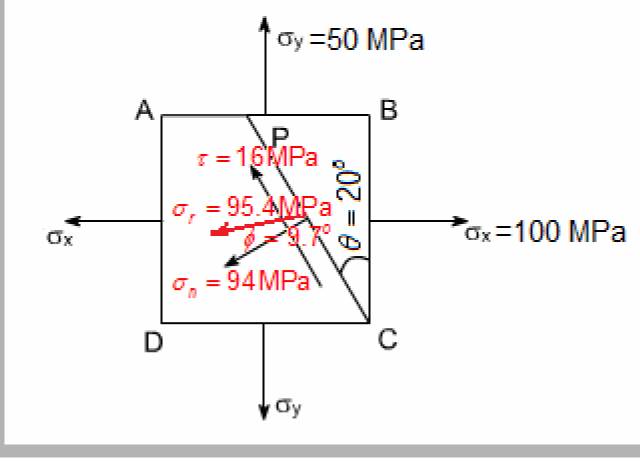
Answer: Given $\sigma_x = 100\text{MPa}\,(\text{tensile}),\ \sigma_y = 50\text{MPa}(\text{tensile})$ and $\theta = 20^0$

$$\text{Normal stress, } (\sigma_n) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta = \frac{100+50}{2} + \frac{100-50}{2}\cos\left(2\times 20^0\right) = 94\text{MPa}$$

$$\text{Shear stress, } (\tau) = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \frac{100-50}{2}\sin\left(2\times 20^0\right) = 16\text{MPa}$$

$$\text{Resultant stress } (\sigma_r) = \sqrt{94^2 + 16^2} = 95.4\text{MPa}$$

$$\text{Therefore angle of obliquity, } (\phi) = \tan^{-1}\left(\frac{\tau}{\sigma_n}\right) = \tan^{-1}\left(\frac{16}{94}\right) = 9.7^0$$



- **We may derive uni-axial stress on oblique plane from**

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

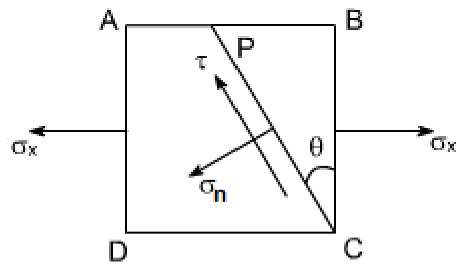
and $$\tau = \frac{\sigma_x - \sigma_y}{2} sin2\theta - \tau_{xy}\cos 2\theta$$

Just put $\sigma_y = 0$ and $\tau_{xy} = 0$

Therefore,

$$\sigma_n = \frac{\sigma_x + 0}{2} + \frac{\sigma_x - 0}{2}\cos 2\theta = \frac{1}{2}\sigma_x\left(1 + \cos 2\theta\right) = \sigma_x \cos^2\theta$$

and $$\tau = \frac{\sigma_x - 0}{2}\sin 2\theta = \frac{\sigma_x}{2}\sin 2\theta$$



2.5 Pure Shear

- **Pure shear is a particular case of bi-axial stress where** $\sigma_x = -\sigma_y$

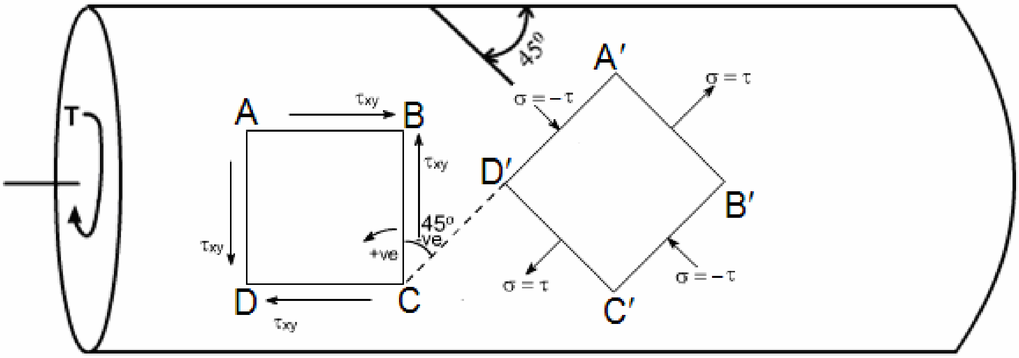
Note: σ_x or σ_y which one is compressive that is immaterial but one should be tensile and other should be compressive and equal magnitude. If $\sigma_x = 100\,\text{MPa}$ then σ_y must be $-100\,\text{MPa}$ otherwise if $\sigma_y = 100\,\text{MPa}$ then σ_x must be $-100\,\text{MPa}$.

- **In case of pure shear on 45° planes**

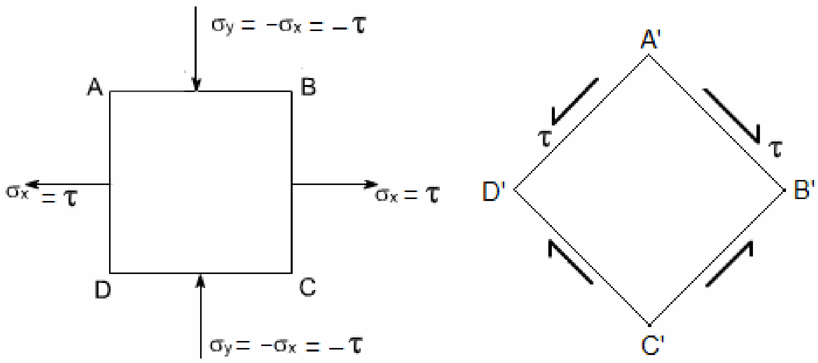
$$\tau_{\max} = \pm\sigma_x \;;\quad \sigma_n = 0 \quad \text{and} \quad \sigma_n' = 0$$

- **We may depict the pure shear in an element by following two ways**

(a) In a torsion member, as shown below, an element ABCD is in pure shear (only shear stress is present in this element) in this member at 45° plane an element $A'B'C'D'$ is also in pure shear where $\sigma_x = -\sigma_y$ but in this element no shear stress is there.



(b) In a bi-axial state of stress a member, as shown below, an element ABCD in pure shear where $\sigma_x = -\sigma_y$ but in this element no shear stress is there and an element $A'B'C'D'$ at 45° plane is also in pure shear (only shear stress is present in this element).



Let us take an example: See the in the Conventional question answer section in this chapter and the question is "*Conventional Question IES-2007*"

2.6 Stress Tensor

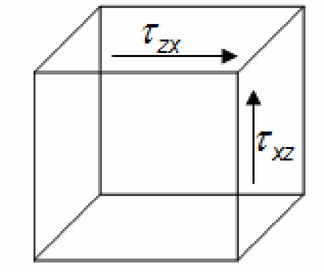
- **State of stress at a point (3-D)**

Stress acts on every surface that passes through the point. We can use three mutually perpendicular planes to describe the stress state at the point, which we approximate as a cube each of the three planes has one normal component & two shear components therefore, 9 components necessary to define stress at a point 3 normal and 6 shear stress.

Therefore, ***we need nine components, to define the state of stress at a point***

$$\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \sigma_y & \tau_{yx} & \tau_{yz} \\ \sigma_z & \tau_{zx} & \tau_{zy} \end{matrix}$$

For cube to be in equilibrium (at rest: not moving, not spinning)



$\tau_{xy} = \tau_{yx}$

$\tau_{xz} = \tau_{zx}$

$\tau_{yz} = \tau_{zy}$

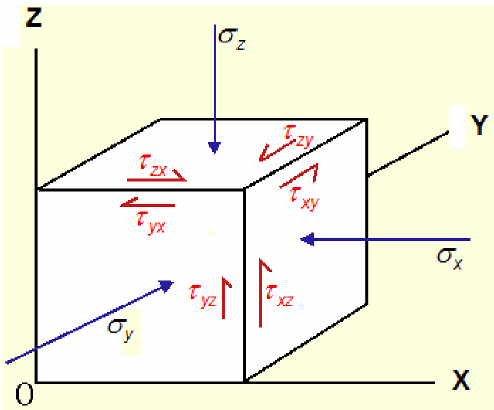
If they don't offset, block spins therefore, ***only six are independent.***

The nine components (six of which are independent) can be written in matrix form

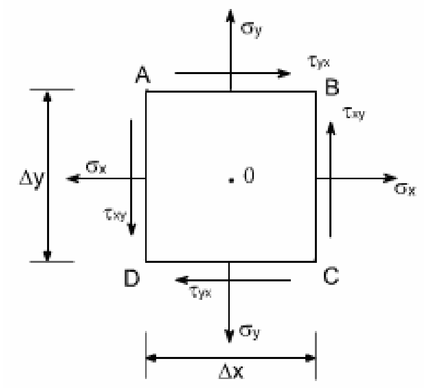
$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \text{ or } \tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

This is the stress tensor

Components on diagonal are normal stresses; off are shear stresses

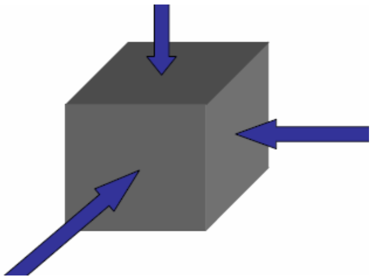


- **State of stress at an element (2-D)**



2.7 Principal stress and Principal plane

- When examining stress at a point, it is possible to choose ***three mutually perpendicular planes*** on which **no shear** stresses exist in three dimensions, one combination of orientations for the three mutually perpendicular planes will cause the shear stresses on all three planes to go to zero *this is the state defined by the principal stresses.*



- *Principal stresses are normal stresses that are orthogonal to each other*
- ***Principal planes*** are the planes across which principal stresses act (faces of the cube) for principal stresses (*shear stresses are zero*)

- **Major Principal Stress**

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- **Minor principal stress**

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- **Position of principal planes**

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

- **Maximum shear stress**

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Let us take an example: In the wall of a cylinder the state of stress is given by, $\sigma_x = 85\text{MPa}$ (compressive), $\sigma_y = 25\text{MPa}$ (tensile) and shear stress $(\tau_{xy}) = 60\text{MPa}$

Calculate the principal planes on which they act. Show it in a figure.

Answer: Given $\sigma_x = -85\text{MPa}, \sigma_y = 25\text{MPa}, \tau_{xy} = 60\text{MPa}$

Major principal stress $(\sigma_1) = \dfrac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$$= \frac{-85+25}{2} + \sqrt{\left(\frac{-85-25}{2}\right)^2 + 60^2} \quad = 51.4\,\text{MPa}$$

Minor principal stress $(\sigma_2) = \dfrac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$$= \frac{-85+25}{2} - \sqrt{\left(\frac{-85-25}{2}\right)^2 + 60^2}$$

$$= -111.4\ \text{MPa} \quad \text{i.e. } 111.4\ \text{MPa (Compressive)}$$

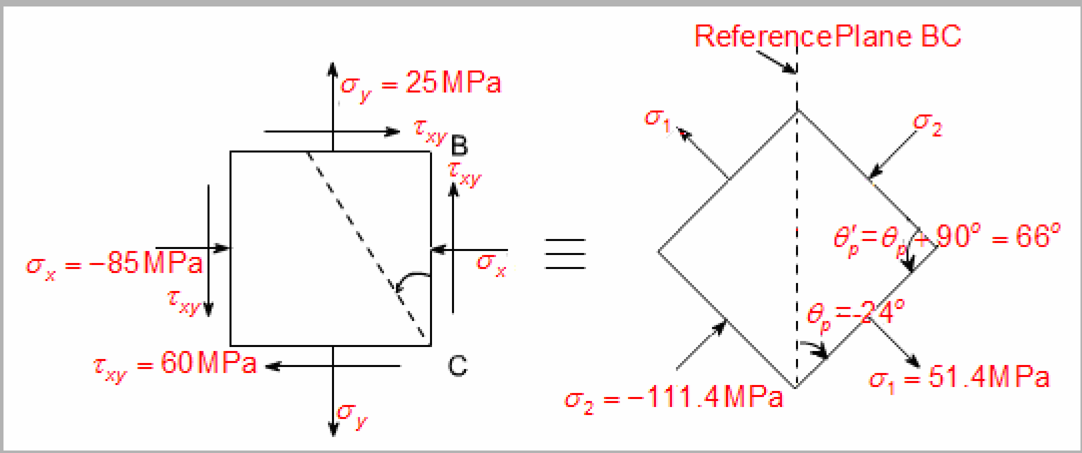
For principal planes

$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\times 60}{-85-25}$$

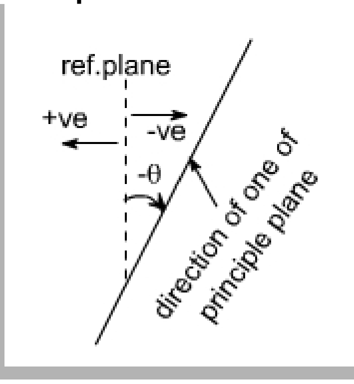
or $\theta_P = -24^0$ it is for σ_1

Complementary plane $\theta_P' = \theta_P + 90 = 66^0$ it is for σ_2

The Figure showing state of stress and principal stresses is given below



The direction of one principle plane and the principle stresses acting on this would be σ_1 when is acting normal to this plane, now the direction of other principal plane would be $90^0 + \theta_p$ because the principal planes are the two mutually perpendicular plane, hence rotate the another plane $90^0 + \theta_p$ in the same direction to get the another plane, now complete the material element as θ_p is negative that means we are measuring the angles in the opposite direction to the reference plane BC. The following figure gives clear idea about negative and positive θ_p.



2.8 Mohr's circle for plane stress

- The transformation equations of plane stress can be represented in a graphical form which is popularly known as ***Mohr's circle.***
- Though the transformation equations are sufficient to get the normal and shear stresses on any plane at a point, with Mohr's circle one can easily visualize their variation with respect to plane orientation θ.

- **Equation of Mohr's circle**

We know that normal stress, $\sigma_n = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

And Tangential stress, $\tau = \dfrac{\sigma_x-\sigma_y}{2} sin 2\theta - \tau_{xy}\cos 2\theta$

Rearranging we get, $\left(\sigma_n - \dfrac{\sigma_x+\sigma_y}{2}\right) = \dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$ ....(i)

and $\tau = \dfrac{\sigma_x-\sigma_y}{2} sin 2\theta - \tau_{xy}\cos 2\theta$ ....(ii)

A little consideration will show that the above two equations are the equations of a circle with σ_n and τ as its coordinates and 2θ as its parameter.

If the parameter 2θ is eliminated from the equations, (i) & (ii) then the significance of them will become clear.

$\sigma_{avg} = \dfrac{\sigma_x+\sigma_y}{2}$ and R = $\sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Or $\left(\sigma_n - \sigma_{avg}\right)^2 + \tau_{xy}^2 = R^2$

It is the equation of a circle with **centre,** $\left(\sigma_{avg}, 0\right)$ *i.e.* $\left(\dfrac{\sigma_x+\sigma_y}{2}, 0\right)$

and **radius,** $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

- ## Construction of Mohr's circle

Convention for drawing

- A τ_{xy} that is clockwise (positive) on a face resides above the σ axis; a τ_{xy} anticlockwise (negative) on a face resides below σ axis.
- Tensile stress will be positive and plotted right of the origin O. Compressive stress will be negative and will be plotted left to the origin O.
- An angle θ on real plane transfers as an angle 2θ on Mohr's circle plane.

We now construct Mohr's circle in the following stress conditions

I. Bi-axial stress when σ_x and σ_y known and $\tau_{xy} = 0$

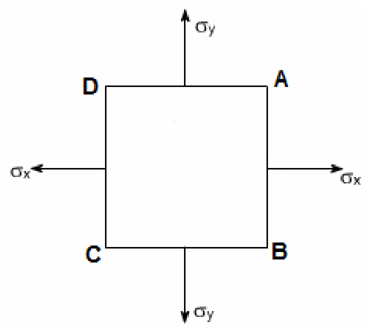
II. Complex state of stress (σ_x, σ_y and τ_{xy} known)

I. *Constant of Mohr's circle for Bi-axial stress* (when only σ_x and σ_y known)

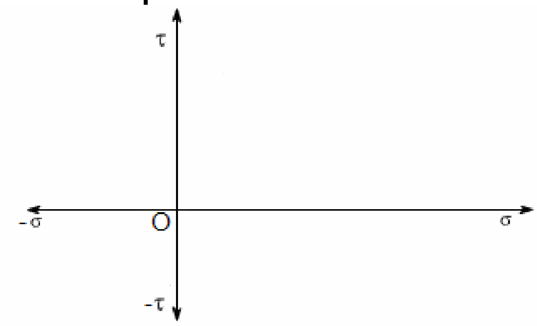
If σ_x and σ_y both are tensile or both compressive sign of σ_x and σ_y will be same and this state of stress is known as " like stresses" if one is tensile and other is compressive sign of σ_x and σ_y will be opposite and this state of stress is known as 'unlike stress'.

- **Construction of Mohr's circle for like stresses (when σ_x and σ_y are same type of stress)**

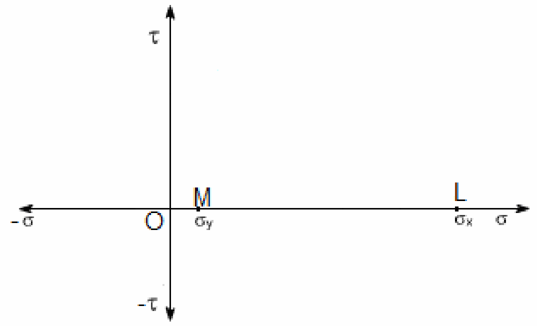
Step-I: Label the element ABCD and draw all stresses.



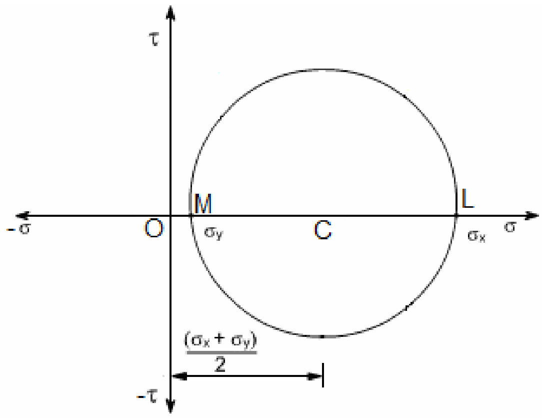
Step-II: Set up axes for the direct stress (as abscissa) i.e., in x-axis and shear stress (as ordinate) i.e. in Y-axis



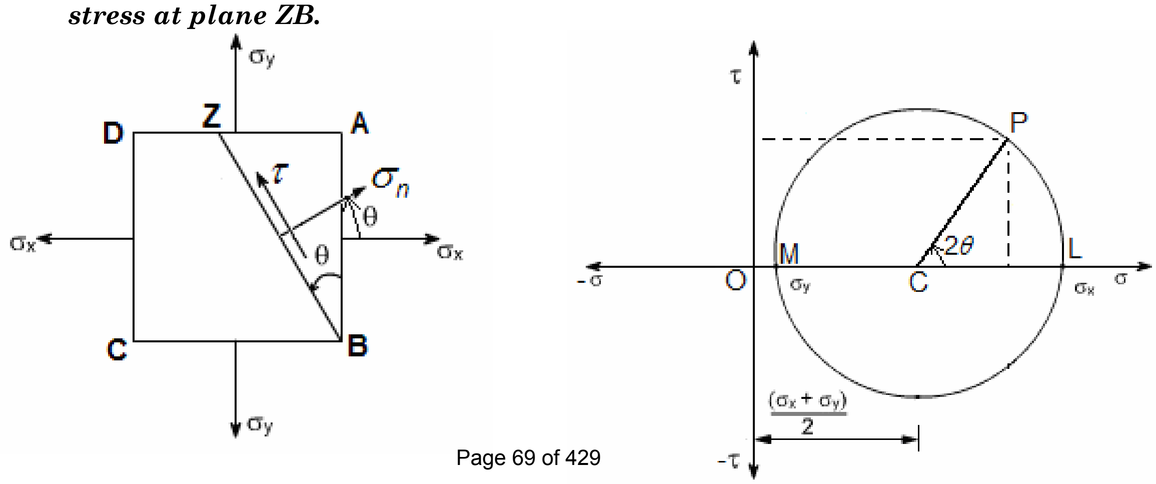
Step-III: Using sign convention and some suitable scale, plot the stresses on two adjacent faces e.g. AB and BC on the graph. Let OL and OM equal to σ_x and σ_y respectively on the axis Oσ.



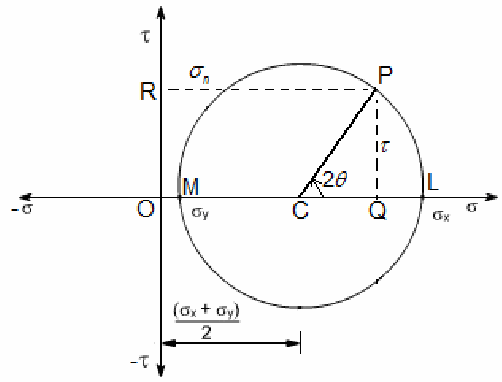
Step-IV: Bisect ML at C. With C as centre and CL or CM as radius, draw a circle. It is the Mohr's circle.



Step-V: At the centre C draw a line CP at an angle 2θ, in the same direction as the normal to the plane makes with the direction of σ_x. ***The point P represents the state of stress at plane ZB.***



Step-VI: ***Calculation,*** Draw a perpendicular PQ and PR where PQ = τ and PR = σ_n



$$OC = \frac{\sigma_x + \sigma_y}{2} \text{ and } MC = CL = CP = \frac{\sigma_x - \sigma_y}{2}$$

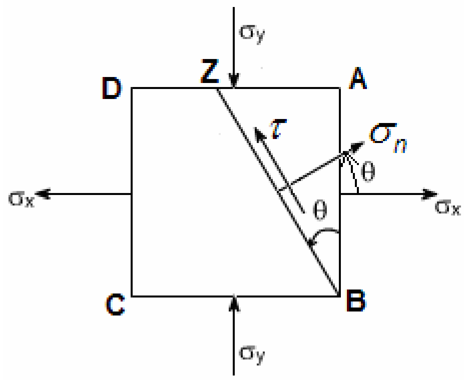
$$PR = \sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta$$

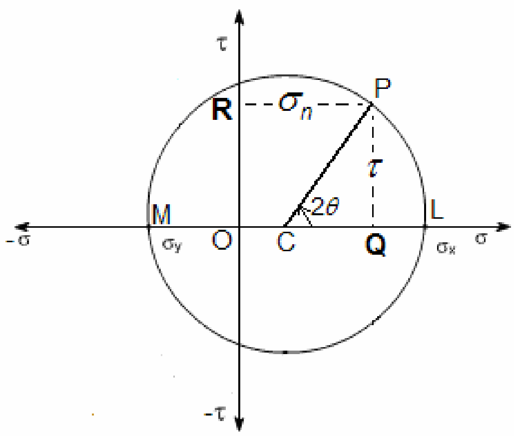
$$PQ = \tau = CP\sin 2\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta$$

[Note: ***In the examination you only draw final figure (which is in Step-V) and follow the procedure step by step so that no mistakes occur.*****]**

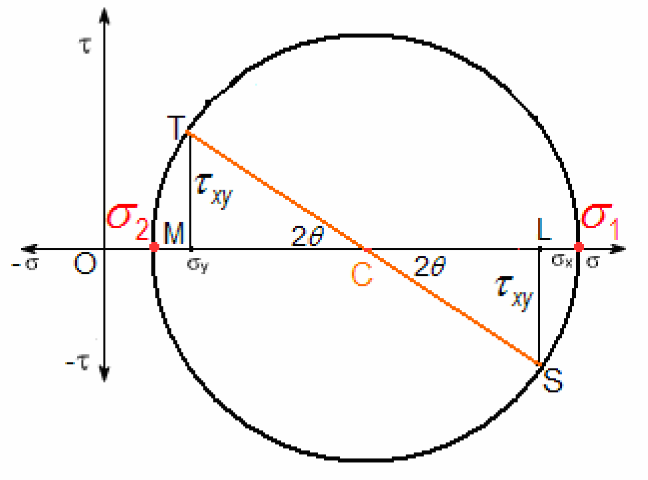
- **Construction of Mohr's circle for unlike stresses (when σ_x and σ_y are opposite in sign)**

  Follow the same steps which we followed for construction for 'like stresses' and finally will get the figure shown below.



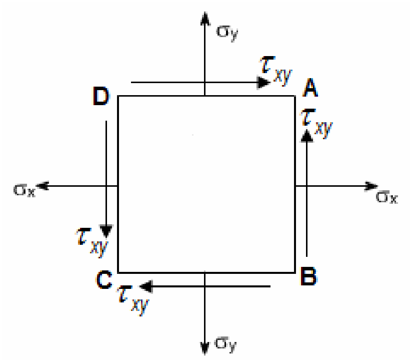


Note: For construction of Mohr's circle for principal stresses when (σ_1 and σ_2 is known) then follow the same steps of Constant of Mohr's circle for Bi-axial stress (when only σ_x and σ_y known) just change the $\sigma_x = \sigma_1$ and $\sigma_y = \sigma_2$

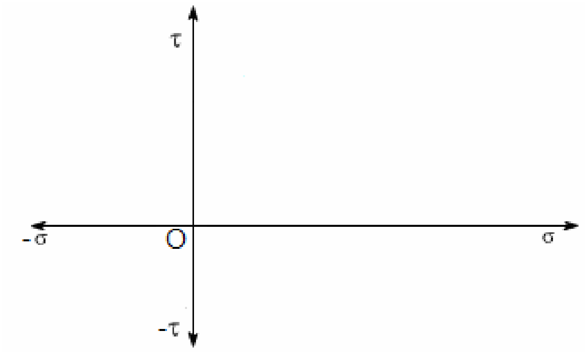


II. ***Construction of Mohr's circle for complex state of stress*** **(σ_x, σ_y and τ_{xy} known)**

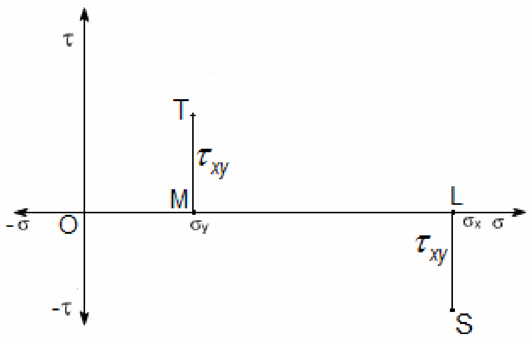
Step-I: Label the element ABCD and draw all stresses.



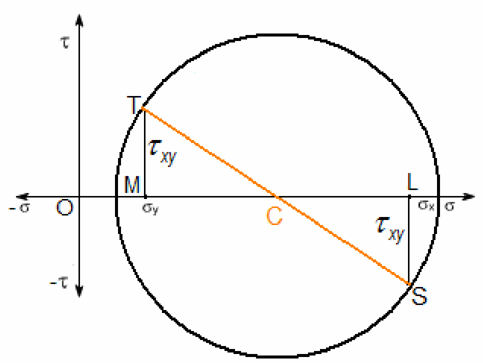
Step-II: Set up axes for the direct stress (as abscissa) i.e., in x-axis and shear stress (as ordinate) i.e. in Y-axis



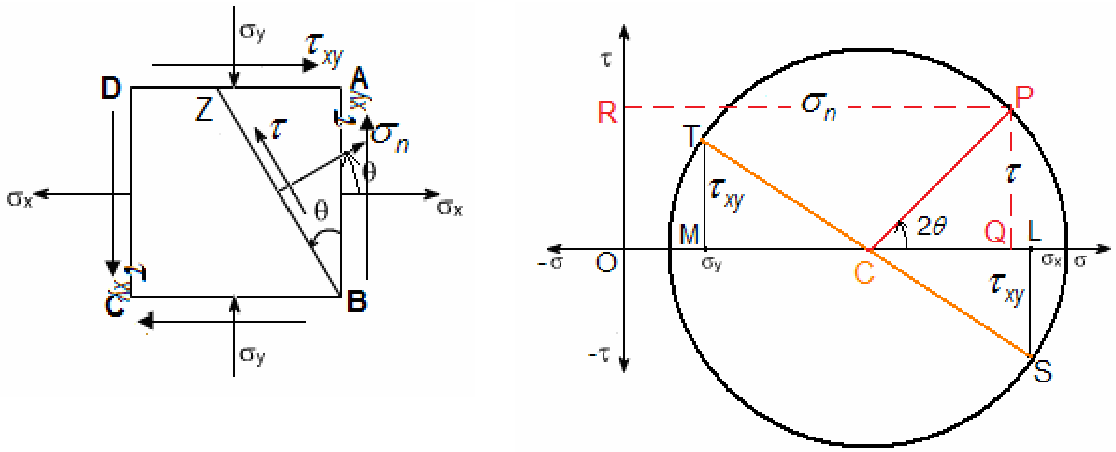
Step-III: Using sign convention and some suitable scale, plot the stresses on two adjacent faces e.g. AB and BC on the graph. Let OL and OM equal to σ_x and σ_y respectively on the axis Oσ. Draw LS perpendicular to $O\sigma$ axis and equal to τ_{xy} .i.e. LS=τ_{xy} . Here LS is downward as τ_{xy} on AB face is (– ive) and draw MT perpendicular to $O\sigma$ axis and equal to τ_{xy} i.e. MT=τ_{xy}. Here MT is upward as τ_{xy} BC face is (+ ive).



Step-IV: Join ST and it will cut $O\sigma$ axis at C. With C as centre and CS or CT as radius, draw circle. It is the Mohr's circle.



Step-V: At the centre draw a line CP at an angle 2θ in the same direction as the normal to the plane makes with the direction of σ_x.



Step-VI: ***Calculation,*** Draw a perpendicular PQ and PR where PQ = τ and PR = σ_n

Centre, OC = $\dfrac{\sigma_x + \sigma_y}{2}$

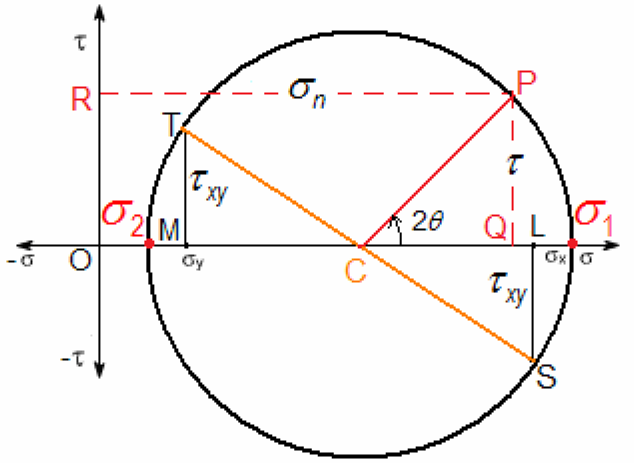
Radius CS = $\sqrt{(CL)^2 + (LS)^2} = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = CT = CP$

$$PR = \sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$PQ = \tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta.$$

[**Note:** ***In the examination you only draw final figure (which is in Step-V) and follow the procedure step by step so that no mistakes occur.***]

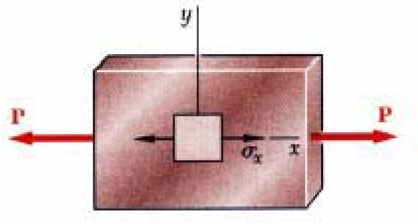
Note: The intersections of $O\sigma$ axis are two principal stresses, as shown below.

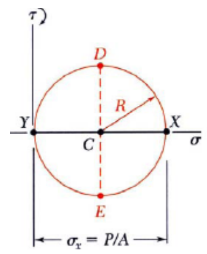


Let us take an example: See the in the Conventional question answer section in this chapter and the question is "Conventional Question IES-2000"

2.9 Mohr's circle for some special cases:

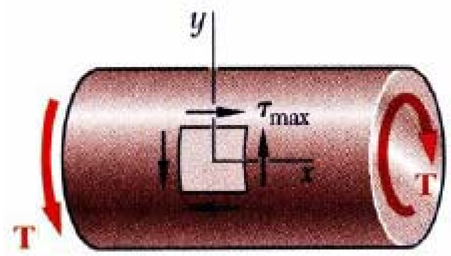
i) Mohr's circle for axial loading:

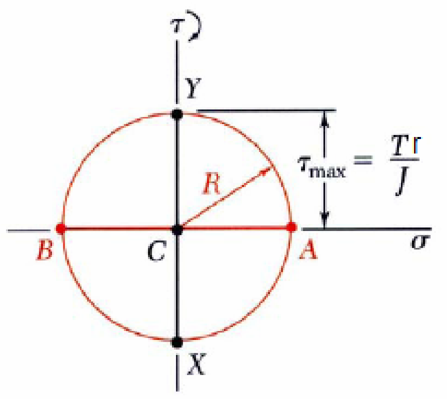




$$\sigma_x = \frac{P}{A};\ \ \sigma_y = \tau_{xy} = 0$$

ii) Mohr's circle for torsional loading:

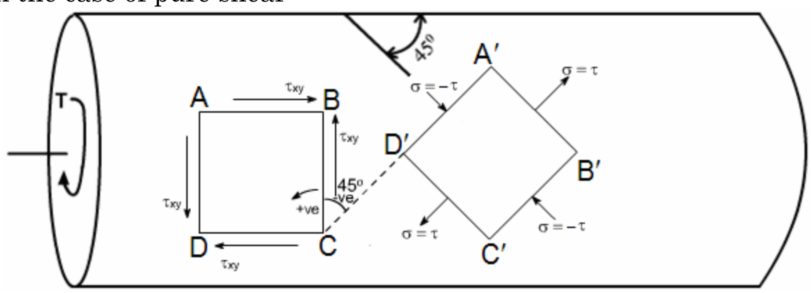


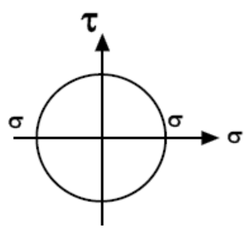


$$\tau_{xy} = \frac{Tr}{J};\ \sigma_x = \sigma_y = 0$$

It is a case of pure shear

iii) In the case of pure shear



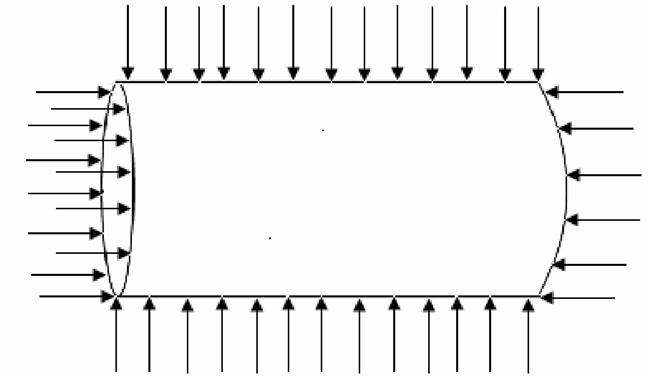


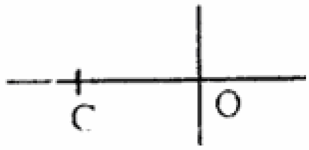
(σ_1 = - σ_2 and σ_3 = 0)

$$\sigma_x = -\sigma_y$$

$$\tau_{\max} = \pm\sigma_x$$

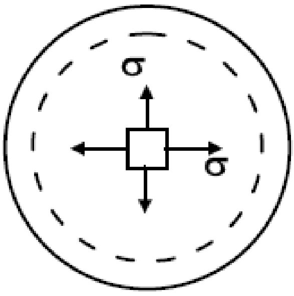
iv) A shaft compressed all round by a hub

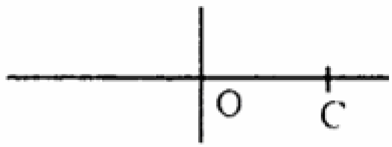




$\sigma_1 = \sigma_2 = \sigma_3$ = Compressive (Pressure)

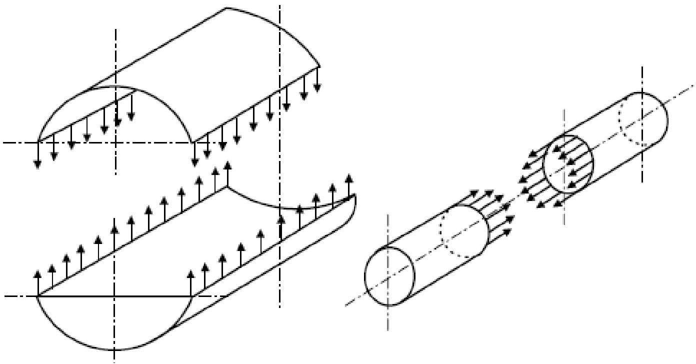
v) Thin spherical shell under internal pressure

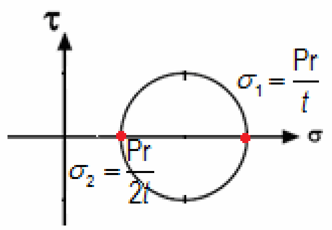




$$\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{pD}{4t} \text{ (tensile)}$$

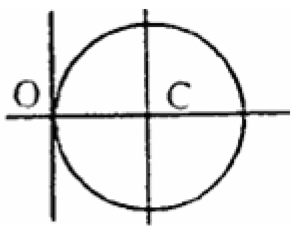
vi) Thin cylinder under pressure





$$\sigma_1 = \frac{pD}{2t} = \frac{pr}{t} \text{ (tensile) and } \sigma_2 = \frac{pd}{4t} = \frac{pr}{2t} \text{ (tensile)}$$

vii) Bending moment applied at the free end of a cantilever



Only bending stress, $\sigma_1 = \dfrac{My}{I}$ and $\sigma_2 = \tau_{xy} = 0$

2.10 Strain

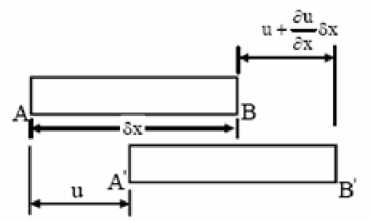
Normal strain

Let us consider an element AB of infinitesimal length δx. After deformation of the actual body if displacement of end A is u, that of end B is $u+\frac{\partial u}{\partial x}.\delta x$. This gives an increase in length of element AB is $\left(u+\frac{\partial u}{\partial x}.\delta x - u\right)=\frac{\partial u}{\partial x}\delta x$ and therefore the strain in x-direction is $\varepsilon_x=\frac{\partial u}{\partial x}$

Similarly, strains in y and z directions are $\varepsilon_y=\frac{\partial \nu}{\partial x}$ and $\varepsilon_z=\frac{\partial w}{\partial z}$.

Therefore, we may write the three normal strain components

$$\varepsilon_x=\frac{\partial u}{\partial x};\qquad \varepsilon_y=\frac{\partial \nu}{\partial y};\qquad \text{and}\qquad \varepsilon_z=\frac{\partial w}{\partial z}.$$



Change in length of an infinitesimal element.

Shear strain

Let us consider an element ABCD in x-y plane and let the displaced position of the element be $A'B'C'D'$.This gives shear strain in x-y plane as $\gamma_{xy}=\propto+\beta$ where $\propto$ is the angle made by the displaced live $B'C'$ with the vertical and β is the angle made by the displaced line $A'D'$ with the horizontal. This gives $\propto=\frac{\frac{\partial u}{\partial x}.\delta y}{\delta y}=\frac{\partial u}{\partial y}$ and $\beta=\frac{\frac{\partial \nu}{\partial x}.\delta x}{\delta x}=\frac{\partial \nu}{\partial x}$

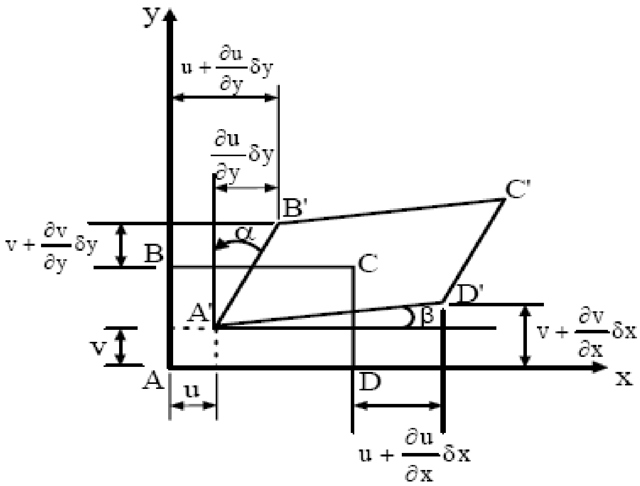
We may therefore write the three shear strain components as

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial \nu}{\partial x};\qquad \gamma_{yz}=\frac{\partial \nu}{\partial z}+\frac{\partial w}{\partial y}\ \text{ and }\ \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

Therefore the state of strain at a point can be completely described by the ***six strain components*** and the strain components in their turns can be completely defined by the displacement components u, ν , and w.

Therefore, the complete strain matrix can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ \upsilon \\ w \end{Bmatrix}$$



Shear strain associated with the distortion of an infinitesimal element.

Strain Tensor

The three normal strain components are

$$\varepsilon_x = \varepsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \varepsilon_y = \varepsilon_{yy} = \frac{\partial \nu}{\partial y} \quad \text{and} \quad \varepsilon_z = \varepsilon_{zz} = \frac{\partial w}{\partial z}.$$

The three shear strain components are

$$\in_{xy} = \frac{\gamma_{xy}}{2} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial \nu}{\partial x}\right); \qquad \in_{yz} = \frac{\gamma_{yz}}{2} = \frac{1}{2}\left(\frac{\partial \nu}{\partial z} + \frac{\partial w}{\partial y}\right) \quad \text{and} \quad \in_{zx} = \frac{\gamma_{zx}}{2} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Therefore the strain tensor is

$$\in_{ij} = \begin{bmatrix} \in_{xx} & \in_{xy} & \in_{xz} \\ \in_{yx} & \in_{yy} & \in_{yz} \\ \in_{zx} & \in_{zy} & \in_{zz} \end{bmatrix} = \begin{bmatrix} \in_{xx} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \in_{yy} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \in_{zz} \end{bmatrix}$$

Constitutive Equation

The constitutive equations relate stresses and strains and in linear elasticity. We know from the Hook's law $(\sigma) = E.\varepsilon$

Where E is modulus of elasticity

It is known that σ_x produces a strain of $\frac{\sigma_x}{\text{E}}$ in x-direction

and Poisson's effect gives $-\mu\frac{\sigma_x}{\text{E}}$ in y-direction **and** $-\mu\frac{\sigma_x}{\text{E}}$ in z-direction.

Therefore we my write the generalized Hook's law as

$$\in_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right], \qquad \in_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_z + \sigma_x\right)\right] \text{ and } \in_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right]$$

It is also known that the shear stress, $\tau = G\gamma$, where G is the shear modulus and γ is shear strain. We may thus write the three strain components as

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \quad \text{and} \quad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

In general each strain is dependent on each stress and we may write

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

$\therefore$ **The number of elastic constant is 36** (For anisotropic materials)

For isotropic material

$$K_{11} = K_{22} = K_{33} = \frac{1}{E}$$

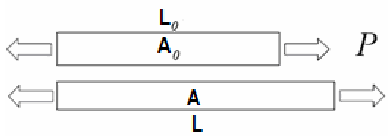
$$K_{44} = K_{55} = K_{66} = \frac{1}{G}$$

$$K_{12} = K_{13} = K_{21} = K_{23} = K_{31} = K_{32} = -\frac{\mu}{E}$$

Rest of all elements in K matrix are zero.

For isotropic material only two independent elastic constant is there say E and G.

- **1-D Strain**



Let us take an example: A rod of cross sectional area A_o is loaded by a tensile force P.

It's stresses $\qquad \sigma_x = \frac{P}{A_o}, \qquad \sigma_y = 0, \quad and \quad \sigma_z = 0$

1-D state of stress or Uni-axial state of stress

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \tau_{ij} = \begin{pmatrix} \tau_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore strain components are

$$\in_x = \frac{\sigma_x}{E}\ ;\ \in_y = -\mu\frac{\sigma_x}{E} = -\mu \in_x\ ;\ \text{and}\ \in_z = -\mu\frac{\sigma_x}{E} = -\mu \in_x$$

1-D state of strain or Uni-axial state of strain

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & -\mu\varepsilon_x & 0 \\ 0 & 0 & -\mu\varepsilon_x \end{pmatrix} = \begin{pmatrix} \frac{\sigma_x}{E} & 0 & 0 \\ 0 & -\mu\frac{\sigma_x}{E} & 0 \\ 0 & 0 & -\mu\frac{\sigma_x}{E} \end{pmatrix} = \begin{pmatrix} p_y & 0 & 0 \\ 0 & q_y & 0 \\ 0 & 0 & q_y \end{pmatrix}$$

- **2-D Strain** $(\sigma_z = 0)$

(i)
$$\in_x = \frac{1}{E}\left[\sigma_x - \mu\sigma_y\right]$$

$$\in_y = \frac{1}{E}\left[\sigma_y - \mu\sigma_x\right]$$

$$\in_z = -\frac{\mu}{E}\left[\sigma_x + \sigma_y\right]$$

[Where, $\in_x, \in_y, \in_z$ are strain component in X, Y, and Z axis respectively]

(ii)
$$\sigma_x = \frac{E}{1-\mu^2}\left[\in_x + \mu \in_y\right]$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\in_y + \mu \in_x\right]$$

- **3-D Strain**

(i)
$$\in_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right]$$

$$\in_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_z + \sigma_x\right)\right]$$

$$\in_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right]$$

(ii)
$$\sigma_x = \frac{E}{(1+\mu)(1-2\mu)}\left[(1-\mu)\in_x + \mu\left(\in_y + \in_z\right)\right]$$

$$\sigma_y = \frac{E}{(1+\mu)(1-2\mu)}\left[(1-\mu)\in_y + \mu(\in_z + \in_x)\right]$$

$$\sigma_z = \frac{E}{(1+\mu)(1-2\mu)}\left[(1-\mu)\in_z + \mu(\in_x + \in_y)\right]$$

Let us take an example: At a point in a loaded member, a state of plane stress exists and the strains are $\varepsilon_x = 270\times10^{-6}$; $\varepsilon_y = -90\times10^{-6}$ and $\varepsilon_{xy} = 360\times10^{-6}$. If the elastic constants E, μ and G are 200 GPa, 0.25 and 80 GPa respectively.

Determine the normal stress σ_x and σ_y and the shear stress τ_{xy} at the point.

Answer: We know that

$$\varepsilon_x = \frac{1}{E}\left\{\sigma_x - \mu\sigma_y\right\}$$

$$.\ \varepsilon_y = \frac{1}{E}\left\{\sigma_y - \mu\sigma_x\right\}$$

$$\varepsilon_{xy} = \frac{\tau_{xy}}{G}$$

This gives $\sigma_x = \frac{E}{1-\mu^2}\left\{\varepsilon_x + \mu\varepsilon_y\right\} = \frac{200\times10^9}{1-0.25^2}\left[+270\times10^{-6} - 0.25\times90\times10^{-6}\right]\text{Pa}$

$= 52.8$ MPa (i.e. tensile)

and $\sigma_y = \frac{E}{1-\mu^2}\left[\varepsilon_y + \mu\varepsilon_x\right]$

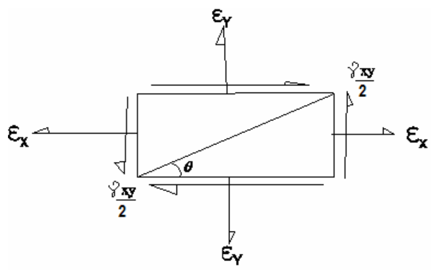
$= \frac{200\times10^9}{1-0.25^2}\left[-90\times10^{-6} + 0.25\times270\times10^{-6}\right]\text{ Pa} = -4.8\text{ MPa}$ (i.e.compressive)

and $\tau_{xy} = \varepsilon_{xy}.G = 360\times10^{-6}\times80\times10^9\text{ Pa} = 28.8\text{MPa}$

2.12 An element subjected to strain components $\in_x, \in_y\ \&\ \frac{\gamma_{xy}}{2}$

Consider an element as shown in the figure given. The strain component In X-direction is $\in_x$, the strain component in Y-direction is $\in_y$ and the shear strain component is γ_{xy}.

Now consider a plane at an angle θ with X- axis in this plane a normal strain $\in_\theta$ and a shear strain γ_θ. Then



- $\in_\theta = \frac{\in_x + \in_y}{2} + \frac{\in_x - \in_y}{2}cos2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$
- $\frac{\gamma_\theta}{2} = -\frac{\in_x - \in_y}{2}sin2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$

We may find principal strain and principal plane for strains in the same process which we followed for stress analysis.

In the principal plane shear strain is zero.

Therefore principal strains are

$$\in_{1,2} = \frac{\in_x + \in_y}{2} \pm \sqrt{\left(\frac{\in_x - \in_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

The angle of principal plane

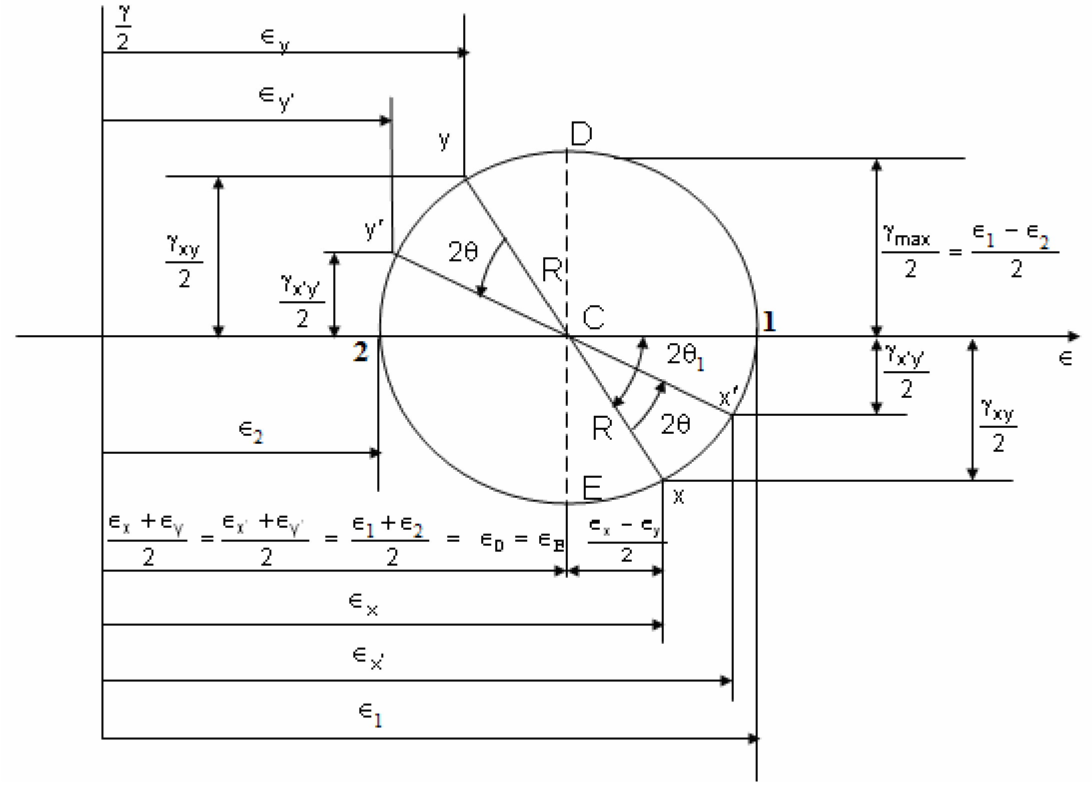
$$\tan 2\theta_p = \frac{\gamma_{xy}}{(\in_x - \in_y)}$$

- Maximum shearing strain is equal to the difference between the 2 principal strains i.e

$$(\gamma_{xy})_{\max} = \in_1 - \in_2$$

Mohr's Circle for circle for Plain Strain

We may draw Mohr's circle for strain following same procedure which we followed for drawing Mohr's circle in stress. Everything will be same and in the place of σ_x write $\in_x$, the place of σ_y write $\in_y$ and in place of τ_{xy} write $\frac{\gamma_{xy}}{2}$.



2.15 Volumetric Strain (Dilation)

- ***Rectangular block,***

$$\frac{\Delta V}{V_0} = \in_x + \in_y + \in_z$$

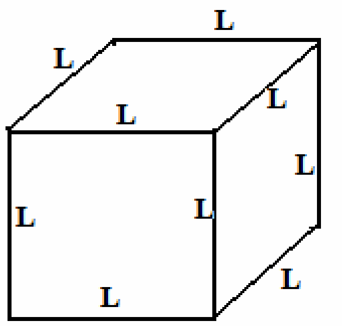
Proof: Volumetric strain

$$\frac{\Delta V}{V_0} = \frac{V - V_o}{V_0}$$

$$= \frac{L(1+\varepsilon_x)\times L(1+\varepsilon_y)\times L(1+\varepsilon_z) - L^3}{L^3}$$

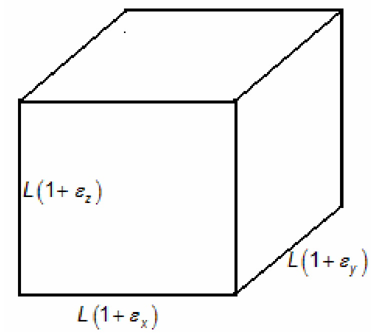
$$= \in_x + \in_y + \in_z$$

(neglecting second and third order term, as very small)



Before deformation,

Volume (V_o) = L^3



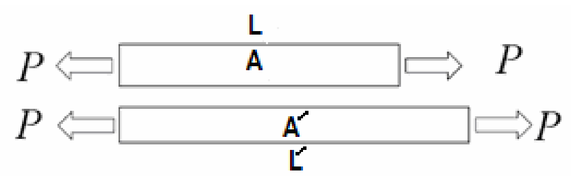
After deformation,

Volume (V)

$= L(1+\varepsilon_x)\times L(1+\varepsilon_y)\times L(1+\varepsilon_z)$

- ***In case of prismatic bar,***

Volumetric strain, $\frac{dv}{v} = \varepsilon(1-2\mu)$



Proof: Before deformation, the volume of the bar, V = A.L

After deformation, the length $(L') = L(1+\varepsilon)$

and the new cross-sectional area $(A') = A(1-\mu\varepsilon)^2$

Therefore now volume $(V') = A'L' = AL(1+\varepsilon)(1-\mu\varepsilon)^2$

$$\therefore \frac{\Delta V}{V} = \frac{V'\text{-}V}{V} = \frac{AL(1+\varepsilon)(1-\mu\varepsilon)^2 - AL}{AL} = \varepsilon(1-2\mu)$$

$$\frac{\Delta V}{V} = \varepsilon(1-2\mu)$$

- ***Thin Cylindrical vessel***

$$\in_1\text{=Longitudinal strain} = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} = \frac{pr}{2Et}[1-2\mu]$$

$$\in_2\text{=Circumferential strain} = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} = \frac{pr}{2Et}[2-\mu]$$

$$\frac{\Delta V}{V_o} = \in_1 + 2\in_2 = \frac{pr}{2Et}[5-4\mu]$$

- ***Thin Spherical vessels***

$$\in = \in_1 = \in_2 = \frac{pr}{2Et}[1-\mu]$$

$$\frac{\Delta V}{V_0} = 3\in = \frac{3pr}{2Et}[1-\mu]$$

- ***In case of pure shear***

$$\sigma_x = -\sigma_y = \tau$$

Therefore

$$\varepsilon_x = \frac{\tau}{E}(1+\mu)$$

$$\varepsilon_y = -\frac{\tau}{E}(1+\mu)$$

$$\varepsilon_z = 0$$

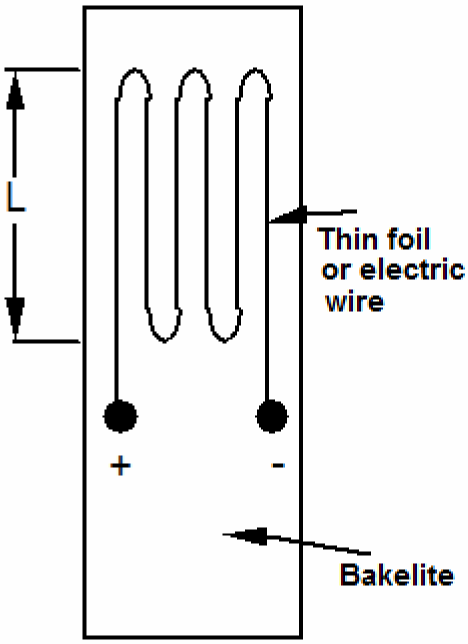
Therefore $\varepsilon_v = \frac{dv}{v} = \varepsilon_x + \varepsilon_y + \varepsilon_z = 0$

2.16 Measurement of Strain

Unlike stress, strain **can** be measured directly. The most common way of measuring strain is by use of the **Strain Gauge.**

Strain Gauge

A strain gage is a simple device, comprising of a thin electric wire attached to an insulating thin backing material such as a bakelite foil. The foil is exposed to the surface of the specimen on which the strain is to be measured. The thin epoxy layer bonds the gauge to the surface and forces the gauge to shorten or elongate as if it were part of the specimen being strained.



STRAIN GAUGE

A change in length of the gauge due to longitudinal strain creates a proportional change in the electric resistance, and since a constant current is maintained in the gauge, a proportional change in voltage. (V = IR).

The voltage can be easily measured, and through calibration, transformed into the change in length of the original gauge length, i.e. the longitudinal strain along the gauge length.

Strain Gauge factor (G.F)

Measured from Bridge voltage

$$G.F. = \frac{\Delta R / R}{\Delta \ell / \ell} = \frac{\Delta R / R}{\varepsilon}$$

Given

Calculated

The strain gauge factor relates a change in resistance with strain.

Strain Rosette

The ***strain rosette*** is a device used to measure the state of strain at a point in a plane.

It comprises ***three or more*** independent strain gauges, each of which is used to read normal strain at the same point but in a different direction.

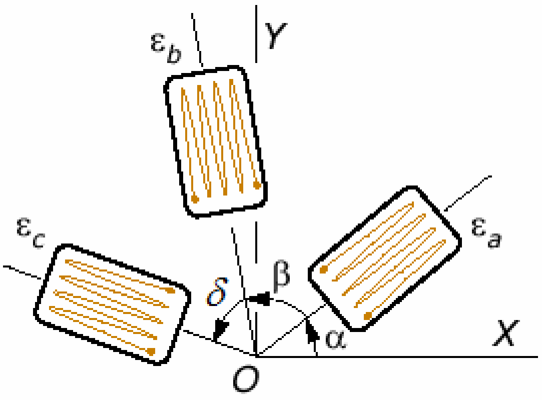
The relative orientation between the three gauges is known as α , β and δ

The three measurements of normal strain provide sufficient information for the determination of the complete state of strain at the measured point in 2-D.

We have to find out $\in_x, \in_y,$ *and* γ_{xy} form measured value $\in_a, \in_b,$ *and* $\in_c$

General arrangement:

The orientation of strain gauges is given in the figure. To relate strain we have to use the following formula.



$$\in_\theta = \frac{\in_x + \in_y}{2} + \frac{\in_x - \in_y}{2} cos2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

We get

$$\in_a = \frac{\in_x + \in_y}{2} + \frac{\in_x - \in_y}{2} cos2\alpha + \frac{\gamma_{xy}}{2} \sin 2\alpha$$

$$\in_b = \frac{\in_x + \in_y}{2} + \frac{\in_x - \in_y}{2} cos2(\alpha + \beta) + \frac{\gamma_{xy}}{2} \sin 2(\alpha + \beta)$$

$$\in_c = \frac{\in_x + \in_y}{2} + \frac{\in_x - \in_y}{2} cos2(\alpha + \beta + \delta) + \frac{\gamma_{xy}}{2} \sin 2(\alpha + \beta + \delta)$$

From this three equations and three unknown we may solve $\in_x$, $\in_y$, *and* γ_{xy}

- **Two standard arrangement of the of the strain rosette are as follows:**

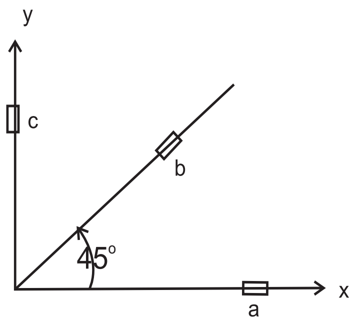
(i) 45° strain rosette or Rectangular strain rosette.

In the general arrangement above, put

$\alpha = 0^o; \ \beta = 45^o \ \ and\ \delta = 45^o$

Putting the value we get

- $\in_a = \in_x$
- $\in_b = \frac{\in_x + \in_x}{2} + \frac{\gamma_{xy}}{2}$
- $\in_c = \in_y$



(ii) 60° strain rosette or Delta strain rosette

In the general arrangement above, put

$\alpha = 0^o; \ \beta = 60^o \ \ and\ \delta = 60^o$

Putting the value we get

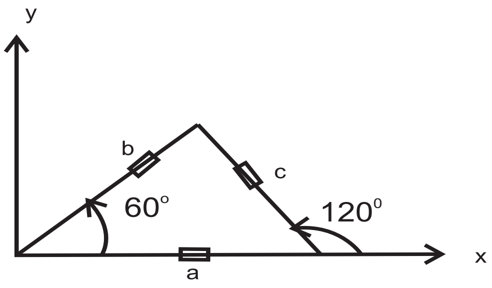
- $\in_a = \in_y$
- $\in_b = \frac{\in_x + 3\in_y}{4} + \frac{\sqrt{3}}{4}\gamma_{xy}$
- $\in_c = \frac{\in_x + 3\in_y}{4} - \frac{\sqrt{3}}{4}\gamma_{xy}$

Solving above three equation we get

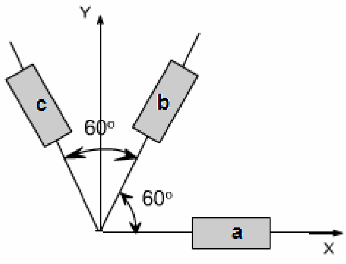
$\in_x = \in_a$

$\in_y = \frac{1}{3}(2.\in_b + 2.\in_c - \in_a)$

$\gamma_{xy} = \frac{2}{\sqrt{3}}(\in_c - \in_b)$



or



OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Stresses due to Pure Shear

GATE-1. A block of steel is loaded by a tangential force on its top surface while the bottom surface is held rigidly. The deformation of the block is due to **[GATE-1992]**

(a) Shear only (b) Bending only (c) Shear and bending (d) Torsion

GATE-1. Ans. (a) It is the definition of shear stress. The force is applied tangentially it is not a point load so you cannot compare it with a cantilever with a point load at its free end.

GATE-2. A shaft subjected to torsion experiences a pure shear stress τ on the surface. The maximum principal stress on the surface which is at 45° to the axis will have a value **[GATE-2003]**

(a) $\tau \cos 45°$ (b) $2\tau \cos 45°$ (c) $\tau \cos^2 45°$ (d) $2\tau \sin 45° \cos 45°$

GATE-2. Ans. (d) $\sigma_n = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

Here $\sigma_x = \sigma_2 = 0,\ \tau_{xy} = \tau,\ \theta = 45°$

GATE-3. The number of components in a stress tensor defining stress at a point in three dimensions is: **[GATE-2002]**

(a) 3 (b) 4 (c) 6 (d) 9

GATE-3. Ans. (d) It is well known that,

$\tau_{xy} = \tau_{yx,}\ \tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$

so that the state of stress at a point is given by six components $\sigma_x, \sigma_y, \sigma_z$ and $\tau_{xy},\ \tau_{yz}, \tau_{zx}$

Principal Stress and Principal Plane

GATE-4. A body is subjected to a pure tensile stress of 100 units. What is the maximum shear produced in the body at some oblique plane due to the above? **[IES-2006]**

(a) 100 units (b) 75 units (c) 50 units (d) 0 unit

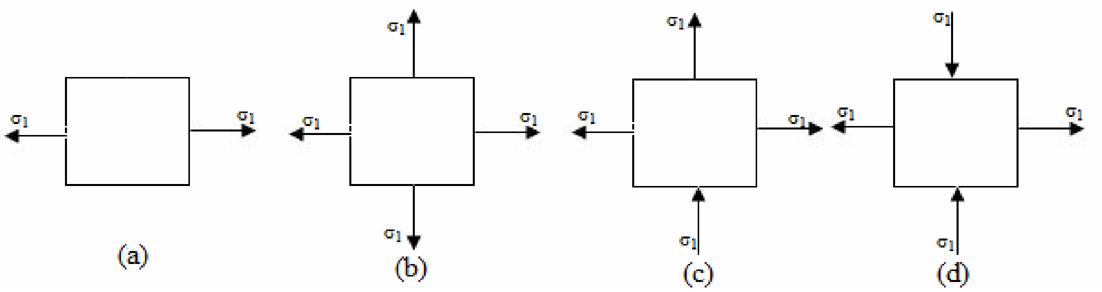
GATE-4. Ans. (c) $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{100 - 0}{2} = 50$ units.

GATE-5. In a strained material one of the principal stresses is twice the other. The maximum shear stress in the same case is τ_{max} .Then, what is the value of the maximum principle stress? **[IES 2007]**

(a) τ_{max} (b) $2\tau_{max}$ (c) $4\tau_{max}$ (d) $8\tau_{max}$

GATE-5. Ans. (c) $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2}$, $\sigma_1 = 2\sigma_2$ or $\tau_{max} = \dfrac{\sigma_2}{2}$ or $\sigma_2 = 2\tau_{max}$ or $\sigma_1 = 2\sigma_2 = 4\tau_{max}$

GATE-6. A material element subjected to a plane state of stress such that the maximum shear stress is equal to the maximum tensile stress, would correspond to **[IAS-1998]**



GATE-6. Ans. (d) $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{\sigma_1 - (-\sigma_1)}{2} = \sigma_1$

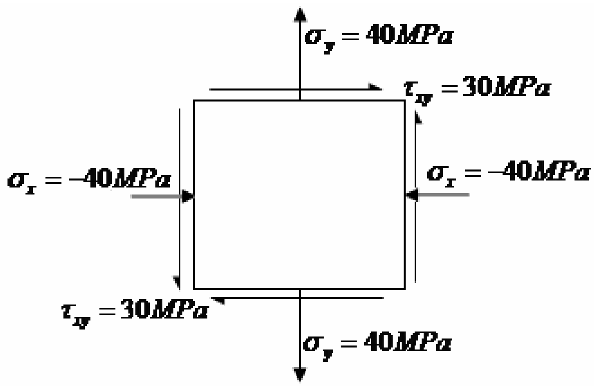
GATE-7. A solid circular shaft is subjected to a maximum shearing stress of 140 MPs. The magnitude of the maximum normal stress developed in the shaft is: [IAS-1995]

(a) 140 MPa (b) 80 MPa (c) 70 MPa (d) 60 MPa

GATE-7. Ans. (a) $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2}$ Maximum normal stress will developed if $\sigma_1 = -\sigma_2 = \sigma$

GATE-8. The state of stress at a point in a loaded member is shown in the figure. The magnitude of maximum shear stress is [1MPa = 10 kg/cm²] [IAS 1994]

(a) 10 MPa (b) 30 MPa (c) 50 MPa (d) 100MPa



GATE-8. Ans. (c) $\tau_{max} = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = \sqrt{\left(\dfrac{-40-40}{2}\right)^2 + 30^2} = 50 \text{ MPa}$

GATE-9. A solid circular shaft of diameter 100 mm is subjected to an axial stress of 50 MPa. It is further subjected to a torque of 10 kNm. The maximum principal stress experienced on the shaft is closest to [GATE-2008]

(a) 41 MPa (b) 82 MPa (c) 164 MPa (d) 204 MPa

GATE-9. Ans. (b) Shear Stress (τ)$= \dfrac{16T}{\pi d^3} = \dfrac{16 \times 10000}{\pi \times (0.1)^3} Pa = 50.93\, MPa$

Maximum principal Stress = $\dfrac{\sigma_b}{2} + \sqrt{\left(\dfrac{\sigma_b}{2}\right)^2 + \tau^2}$ =82 MPa

GATE-10. In a bi-axial stress problem, the stresses in x and y directions are (σ_x = 200 MPa and σ_y =100 MPa. The maximum principal stress in MPa, is: [GATE-2000]

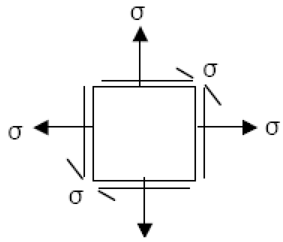
(a) 50 (b) 100 (c) 150 (d) 200

GATE-10. Ans. (d) $\sigma_1 = \dfrac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ if $\tau_{xy} = 0$

$$=\frac{\sigma_x+\sigma_y}{2}+\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2}=\sigma_x$$

GATE-11. The maximum principle stress for the stress state shown in the figure is

(a) σ (b) 2 σ
(c) 3 σ (d) 1.5 σ



[GATE-2001]

GATE-11. Ans. (b) $\sigma_x=\sigma,\quad \sigma_y=\sigma,\quad \tau_{xy}=\sigma$

$$\therefore(\sigma_1)_{max}=\frac{\sigma_x+\sigma_y}{2}+\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}=\frac{\sigma+\sigma}{2}+\sqrt{(0)^2+\sigma^2}=2\sigma$$

GATE-12. The normal stresses at a point are σx = 10 MPa and, σy = 2 MPa; the shear stress at this point is 4MPa. The maximum principal stress at this point is: [GATE-1998]

(a) 16 MPa (b) 14 MPa (c) 11 MPa (d) 10 MPa

GATE-12. Ans. (c) $\sigma_1=\frac{\sigma_x+\sigma_y}{2}+\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\ =\frac{10+2}{2}+\sqrt{\left(\frac{10-2}{2}\right)^2+4^2}=11.66\text{ MPa}$

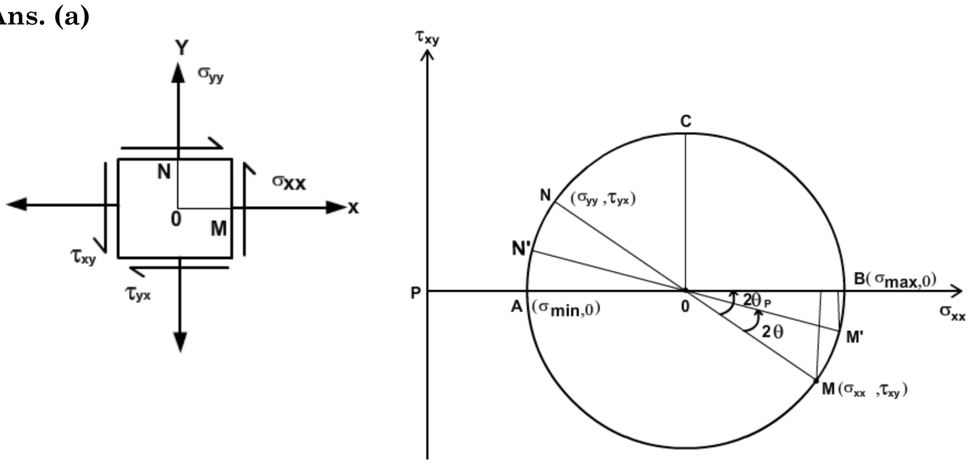
GATE-13. In a Mohr's circle, the radius of the circle is taken as: [IES-2006; GATE-1993]

(a) $\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\left(\tau_{xy}\right)^2}$ (b) $\sqrt{\frac{\left(\sigma_x-\sigma_y\right)^2}{2}+\left(\tau_{xy}\right)^2}$

(c) $\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2-\left(\tau_{xy}\right)^2}$ (d) $\sqrt{\left(\sigma_x-\sigma_y\right)^2+\left(\tau_{xy}\right)^2}$

Where, σ_x and σ_y are normal stresses along x and y directions respectively and τ_{xy} is the shear stress.

GATE-13. Ans. (a)



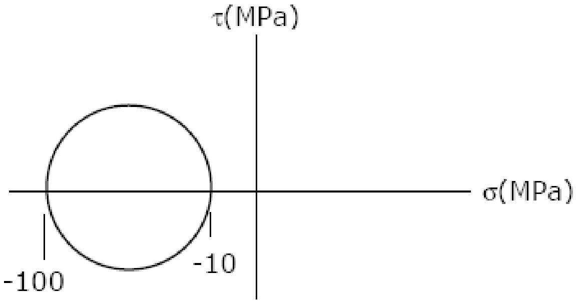
GATE-14. A two dimensional fluid element rotates like a rigid body. At a point within the element, the pressure is 1 unit. Radius of the Mohr's circle, characterizing the state of stress at that point, is: [GATE-2008]

(a) 0.5 unit (b) 0 unit (c) 1 unit (d) 2 units

GATE-14. Ans. (b)

GATE-15. The Mohr's circle of plane stress for a point in a body is shown. The design is to be done on the basis of the maximum shear stress theory for yielding. Then, yielding will just begin if the designer chooses a ductile material whose yield strength is:

(a) 45 MPa (b) 50 MPa
(c) 90 MPa (d) 100 MPa



[GATE-2005]

GATE-15. Ans. (c)

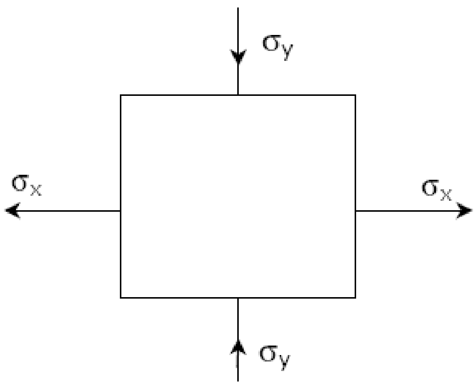
Given $\sigma_1 = -10$ MPa, $\sigma_2 = -100$ MPa

Maximum shear stress theory give $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{\sigma_y}{2}$

or $\sigma_1 - \sigma_2 = \sigma_y$ $\Rightarrow \sigma_y = -10 - (-100) = 90$ MPa

GATE-16. The figure shows the state of stress at a certain point in a stressed body. The magnitudes of normal stresses in the x and y direction are 100MPa and 20 MPa respectively. The radius of Mohr's stress circle representing this state of stress is:

(a) 120 (b) 80
(c) 60 (d) 40



[GATE-2004]

GATE-16. Ans. (c)

$\sigma_x = 100$ MPa, $\sigma_y = -20$ MPa

Radius of Mohr's circle $= \dfrac{\sigma_x - \sigma_y}{2} = \dfrac{100 - (-20)}{2} = 60$

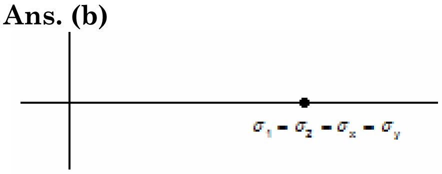
Data for Q17–Q18 are given below. Solve the problems and choose correct answers. **[GATE-2003]**

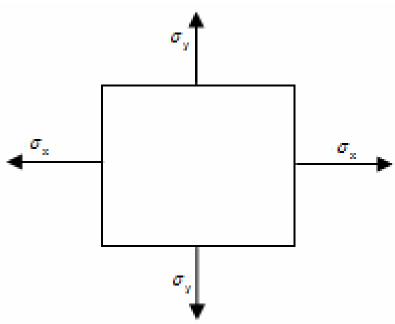
The state of stress at a point "P" in a two dimensional loading is such that the Mohr's circle is a point located at 175 MPa on the positive normal stress axis.

GATE-17. Determine the maximum and minimum principal stresses respectively from the Mohr's circle

(a) + 175 MPa, –175MPa (b) +175 MPa, +175 MPa
(c) 0, –175 MPa (d) 0, 0

GATE-17. Ans. (b)





$\sigma_1 = \sigma_2 = \sigma_x = \sigma_y = +175$ MPa

GATE-18. Determine the directions of maximum and minimum principal stresses at the point "P" from the Mohr's circle **[GATE-2003]**

(a) 0, 90° (b) 90°, 0 (c) 45°, 135° (d) All directions

GATE-18. Ans. (d) From the Mohr's circle it will give all directions.

Principal strains

GATE-19. If the two principal strains at a point are 1000 × 10^{-6} and -600 × 10^{-6}, then the maximum shear strain is: [GATE-1996]

(a) 800 × 10^{-6} (b) 500 × 10^{-6} (c) 1600 × 10^{-6} (d) 200 × 10^{-6}

GATE-19. Ans. (c) Shear strain $e_{max} - e_{min} = \{1000 - (-600)\} \times 10^{-6} = 1600 \times 10^{-6}$

Previous 20-Years IES Questions

Stresses due to Pure Shear

IES-1. If a prismatic bar be subjected to an axial tensile stress σ, then shear stress induced on a plane inclined at θ with the axis will be: [IES-1992]

$(a) \frac{\sigma}{2}\sin 2\theta$ $(b) \frac{\sigma}{2}\cos 2\theta$ $(c) \frac{\sigma}{2}\cos^2\theta$ $(d) \frac{\sigma}{2}\sin^2\theta$

IES-1. Ans. (a)

IES-2. In the case of bi-axial state of normal stresses, the normal stress on 45° plane is equal to [IES-1992]

(a) The sum of the normal stresses (b) Difference of the normal stresses
(c) Half the sum of the normal stresses (d) Half the difference of the normal stresses

IES-2. Ans. (c) $\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

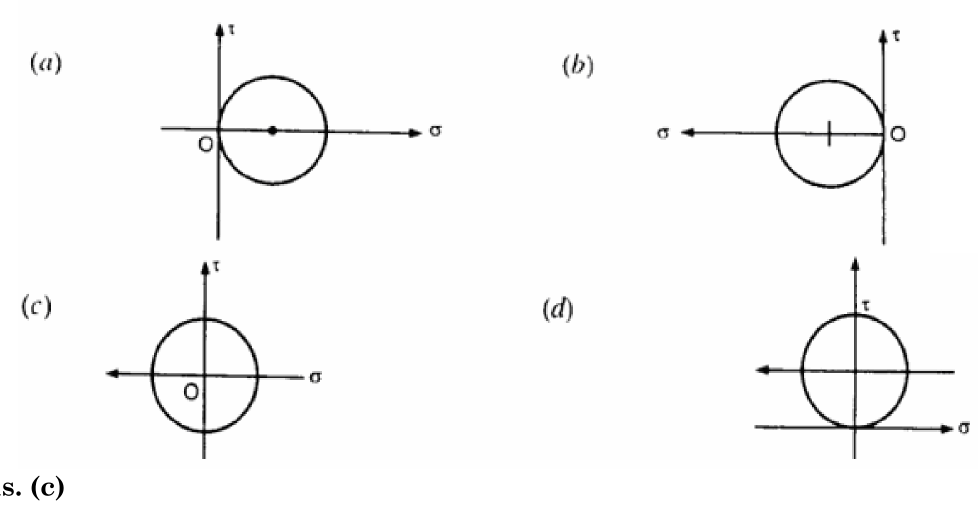
$\text{At } \theta = 45^\circ \text{ and } \tau_{xy} = 0; \quad \sigma_n = \frac{\sigma_x + \sigma_y}{2}$

IES-3. In a two-dimensional problem, the state of pure shear at a point is characterized by [IES-2001]

(a) $\varepsilon_x = \varepsilon_y \text{ and } \gamma_{xy} = 0$ (b) $\varepsilon_x = -\varepsilon_y \text{ and } \gamma_{xy} \neq 0$
(c) $\varepsilon_x = 2\varepsilon_y \text{ and } \gamma_{xy} \neq 0$ (d) $\varepsilon_x = 0.5\varepsilon_y \text{ and } \gamma_{xy} = 0$

IES-3. Ans. (b)

IES-4. Which one of the following Mohr's circles represents the state of pure shear? [IES-2000]



IES-4. Ans. (c)

IES-5. For the state of stress of pure shear τ the strain energy stored per unit volume in the elastic, homogeneous isotropic material having elastic constants E and ν will be: [IES-1998]

(a) $\frac{\tau^2}{E}(1+\nu)$ (b) $\frac{\tau^2}{2E}(1+\nu)$ (c) $\frac{2\tau^2}{E}(1+\nu)$ (d) $\frac{\tau^2}{2E}(2+\nu)$

IES-5. Ans. (a) $\sigma_1 = \tau, \quad \sigma_2 = -\tau, \quad \sigma_3 = 0$

$$U = \frac{1}{2E}\left[\tau^2 + (-\tau)^2 - 2\mu\tau(-\tau)\right]V = \frac{1+\mu}{E}\tau^2 V$$

IES-6. Assertion (A): If the state at a point is pure shear, then the principal planes through that point making an angle of 45° with plane of shearing stress carries principal stresses whose magnitude is equal to that of shearing stress.
Reason (R): Complementary shear stresses are equal in magnitude, but opposite in direction. [IES-1996]

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-6. Ans. (b)

IES-7. Assertion (A): Circular shafts made of brittle material fail along a helicoidally surface inclined at 45° to the axis (artery point) when subjected to twisting moment. [IES-1995]
Reason (R): The state of pure shear caused by torsion of the shaft is equivalent to one of tension at 45° to the shaft axis and equal compression in the perpendicular direction.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-7. Ans. (a) Both A and R are true and R is correct explanation for A.

IES-8. A state of pure shear in a biaxial state of stress is given by [IES-1994]

(a) $\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}$ (b) $\begin{pmatrix}\sigma_1 & 0\\ 0 & -\sigma_1\end{pmatrix}$ (c) $\begin{pmatrix}\sigma_x & \tau_{xy}\\ \tau_{yx} & \sigma_y\end{pmatrix}$ (d) None of the above

IES-8. Ans. (b) $\sigma_1 = \tau, \quad \sigma_2 = -\tau, \quad \sigma_3 = 0$

IES-9. The state of plane stress in a plate of 100 mm thickness is given as [IES-2000]
σ_{xx} = 100 N/mm², σ_{yy} = 200 N/mm², Young's modulus = 300 N/mm², Poisson's ratio = 0.3. The stress developed in the direction of thickness is:

(a) Zero (b) 90 N/mm² (c) 100 N/mm² (d) 200 N/mm²

IES-9. Ans. (a)

IES-10. The state of plane stress at a point is described by $\sigma_x = \sigma_y = \sigma$ and $\tau_{xy} = 0$. The normal stress on the plane inclined at 45° to the x-plane will be: [IES-1998]

(a) σ (b) $\sqrt{2}\sigma$ (c) $\sqrt{3}\sigma$ (d) 2σ

IES-10. Ans. (a) $\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

IES-11. Consider the following statements: [IES-1996, 1998]
State of stress in two dimensions at a point in a loaded component can be completely specified by indicating the normal and shear stresses on

1. **A plane containing the point**
2. **Any two planes passing through the point**
3. **Two mutually perpendicular planes passing through the point**

Of these statements

(a) 1, and 3 are correct (b) 2 alone is correct

(c) 1 alone is correct (d) 3 alone is correct

IES-11. Ans. (d)

Principal Stress and Principal Plane

IES-12. A body is subjected to a pure tensile stress of 100 units. What is the maximum shear produced in the body at some oblique plane due to the above? [IES-2006]

(a) 100 units (b) 75 units (c) 50 units (d) 0 unit

IES-12. Ans. (c) $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{100-0}{2} = 50$ units.

IES-13. In a strained material one of the principal stresses is twice the other. The maximum shear stress in the same case is τ_{max}. Then, what is the value of the maximum principle stress? [IES 2007]

(a) τ_{max} (b) $2\tau_{max}$ (c) $4\tau_{max}$ (d) $8\tau_{max}$

IES-13. Ans. (c) $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2}$, $\sigma_1 = 2\sigma_2$ or $\tau_{max} = \dfrac{\sigma_2}{2}$ or $\sigma_2 = 2\tau_{max}$ or $\sigma_1 = 2\sigma_2 = 4\tau_{max}$

IES-14. In a strained material, normal stresses on two mutually perpendicular planes are σ_x and σ_y (both alike) accompanied by a shear stress τ_{xy} One of the principal stresses will be zero, only if [IES-2006]

(a) $\tau_{xy} = \dfrac{\sigma_x \times \sigma_y}{2}$ (b) $\tau_{xy} = \sigma_x \times \sigma_y$ (c) $\tau_{xy} = \sqrt{\sigma_x \times \sigma_y}$ (d) $\tau_{xy} = \sqrt{\sigma_x^2 + \sigma_y^2}$

IES-14. Ans. (c) $\sigma_{1,2} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

if $\sigma_2 = 0 \quad \Rightarrow \dfrac{\sigma_x + \sigma_y}{2} = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$

or $\left(\dfrac{\sigma_x + \sigma_y}{2}\right)^2 = \left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}$ or $\tau_{xy} = \sqrt{\sigma_x \times \sigma_y}$

IES-15. The principal stresses σ_1, σ_2 and σ_3 at a point respectively are 80 MPa, 30 MPa and –40 MPa. The maximum shear stress is: [IES-2001]

(a) 25 MPa (b) 35 MPa (c) 55 MPa (d) 60 MPa

IES-15. Ans. (d) $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{80-(-40)}{2} = 60$ MPa

IES-16. Plane stress at a point in a body is defined by principal stresses 3σ and σ. The ratio of the normal stress to the maximum shear stresses on the plane of maximum shear stress is: [IES-2000]

(a) 1 (b) 2 (c) 3 (d) 4

IES-16. Ans. (b) $\tan 2\theta = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y} \Rightarrow \theta = 0$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{3\sigma - \sigma}{2} = \sigma$$

Major principal stress on the plane of maximum shear = $\sigma_1 = \dfrac{3\sigma + \sigma}{2} = 2\sigma$

IES-17. Principal stresses at a point in plane stressed element are $\sigma_x = \sigma_y = 500\,\text{kg/cm}^2$. Normal stress on the plane inclined at 45° to x-axis will be: [IES-1993]

(a) 0 (b) 500 kg/cm² (c) 707 kg/cm² (d) 1000 kg/cm²

IES-17. Ans. (b) When stresses are alike, then normal stress σ_n on plane inclined at angle 45° is

$$\sigma_n = \sigma_y \cos^2\theta + \sigma_x \sin^2\theta = \sigma_y\left(\frac{1}{\sqrt{2}}\right)^2 + \sigma_x\left(\frac{1}{\sqrt{2}}\right)^2 = 500\left[\frac{1}{2}+\frac{1}{2}\right] = 500\,\text{kg/cm}^2$$

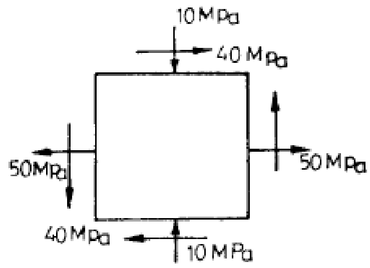
IES-18. If the principal stresses corresponding to a two-dimensional state of stress are σ_1 and σ_2 is greater than σ_2 and both are tensile, then which one of the following would be the correct criterion for failure by yielding, according to the maximum shear stress criterion? [IES-1993]

$(a)\dfrac{(\sigma_1-\sigma_2)}{2} = \pm\dfrac{\sigma_{yp}}{2}$ $(b)\dfrac{\sigma_1}{2} = \pm\dfrac{\sigma_{yp}}{2}$ $(c)\dfrac{\sigma_2}{2} = \pm\dfrac{\sigma_{yp}}{2}$ $(d)\sigma_1 = \pm 2\sigma_{yp}$

IES-18. Ans. (a)

IES-19. For the state of plane stress. Shown the maximum and minimum principal stresses are:

(a) 60 MPa and 30 MPa
(b) 50 MPa and 10 MPa
(c) 40 MPa and 20 MPa
(d) 70 MPa and 30 MPa



[IES-1992]

IES-19. Ans. (d) $\sigma_{1,2} = \dfrac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$

$$\sigma_{1,2} = \frac{50+(-10)}{2} \pm \sqrt{\left(\frac{50+10}{2}\right)^2 + 40^2}$$

$$\sigma_{max} = 70 \text{ and } \sigma_{min} = -30$$

IES-20. Normal stresses of equal magnitude p, but of opposite signs, act at a point of a strained material in perpendicular direction. What is the magnitude of the resultant normal stress on a plane inclined at 45° to the applied stresses? [IES-2005]

(a) 2 p (b) p/2 (c) p/4 (d) Zero

IES-20. Ans. (d) $\sigma_x = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta$

$$\sigma_n = \frac{P-P}{2} + \frac{P+P}{2}\cos 2\times 45 = 0$$

IES-21. A plane stressed element is subjected to the state of stress given by $\sigma_x = \tau_{xy} = 100\,\text{kgf/cm}^2$ and $\sigma_y = 0$. Maximum shear stress in the element is equal to [IES-1997]

(a) $50\sqrt{3}\,\text{kgf/cm}^2$ (b) $100\,\text{kgf/cm}^2$ (c) $50\sqrt{5}\,\text{kgf/cm}^2$ (d) $150\,\text{kgf/cm}^2$

IES-21. Ans. (c) $(\sigma)_{1,2} = \dfrac{\sigma_x + 0}{2} \pm \sqrt{\left(\dfrac{\sigma_x + 0}{2}\right)^2 + \tau_{xy}^2} = 50 \mp 50\sqrt{5}$

Maximum shear stress = $\dfrac{(\sigma)_1 - (\sigma)_2}{2} = 50\sqrt{5}$

IES-22. Match List I with List II and select the correct answer, using the codes given below the lists: **[IES-1995]**

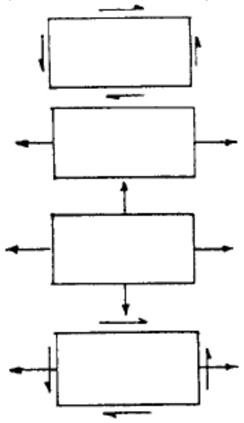
List I(State of stress)

A.

B.

C.

D.



List II(Kind of loading)

1. Combined bending and torsion of circular shaft.
2. Torsion of circular shaft.
3. Thin cylinder subjected to internal pressure.
4. Tie bar subjected to tensile force.

Codes:	A	B	C	D		A	B	C	D
(a)	1	2	3	4	(b)	2	3	4	1
(c)	2	4	3	1	(d)	3	4	1	2

IES-22. Ans. (c)

Mohr's circle

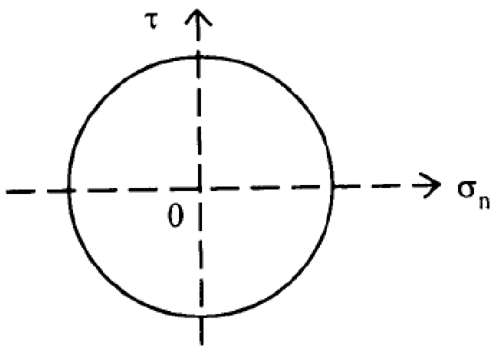
IES-23. Consider the Mohr's circle shown above:
What is the state of stress represented by this circle?

(a) $\sigma_x = \sigma_y \neq 0, \tau_{xy} = 0$

(b) $\sigma_x + \sigma_y = 0, \tau_{xy} \neq 0$

(c) $\sigma_x = 0, \sigma_y = \tau_{xy} \neq 0$

(d) $\sigma_x \neq 0, \sigma_y = \tau_{xy} = 0$



[IES-2008]

IES-23. Ans. (b) It is a case of pure shear. Just put $\sigma_1 = -\sigma_2$

IES-24. For a general two dimensional stress system, what are the coordinates of the centre of Mohr's circle? [II

(a) $\dfrac{\sigma_x - \sigma_y}{2}, 0$ (b) $0, \dfrac{\sigma_x + \sigma_y}{2}$ (c) $\dfrac{\sigma_x + \sigma_y}{2}, 0$ (d) $0, \dfrac{\sigma_x - \sigma_y}{2}$

IES-24. Ans. (c)

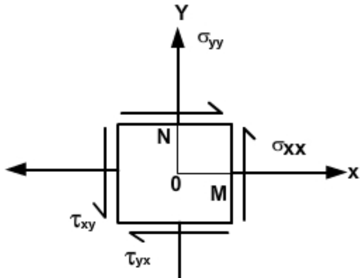
IES-25. In a Mohr's circle, the radius of the circle is taken as: **[IES-2006; GATE-1993]**

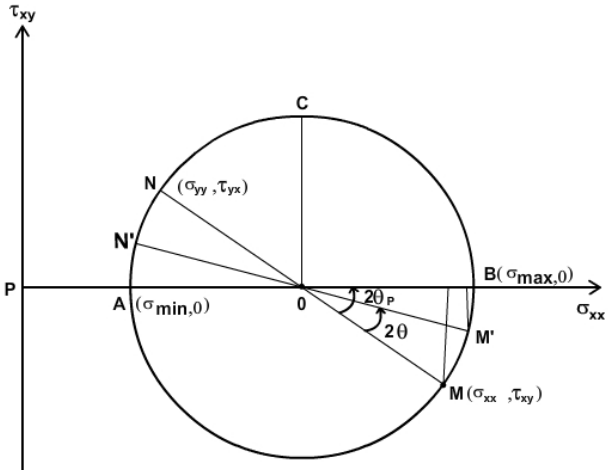
(a) $\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$ (b) $\sqrt{\dfrac{(\sigma_x - \sigma_y)^2}{2} + (\tau_{xy})^2}$

(c) $\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 - (\tau_{xy})^2}$ (d) $\sqrt{(\sigma_x - \sigma_y)^2 + (\tau_{xy})^2}$

Where, σ_x and σ_y are normal stresses along x and y directions respectively and τ_{xy} is the shear stress.

IES-25. Ans. (a)



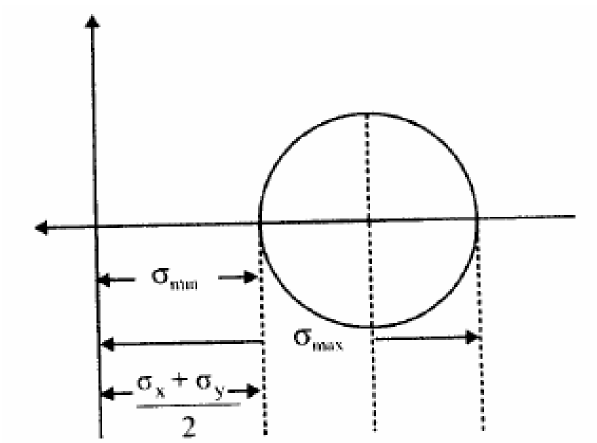


IES-26. **Maximum shear stress in a Mohr's Circle** **[IES- 2008]**

(a) Is equal to radius of Mohr's circle (b) Is greater than radius of Mohr's circle

(c) Is less than radius of Mohr's circle (d) Could be any of the above

IES-26. Ans. (a)



$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\right)^2$$

$\therefore$ Radius of the Mohr Circle

$$\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\therefore \ \sigma_t = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\Rightarrow \tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = r \qquad \Rightarrow \tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

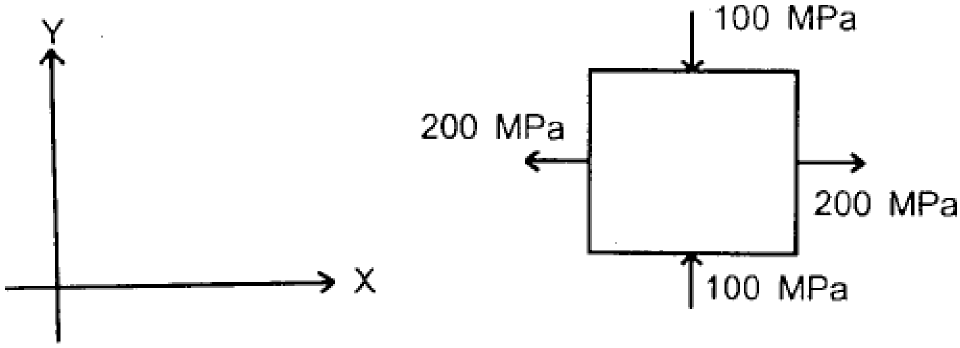
IES-27. **At a point in two-dimensional stress system σ_x = 100 N/mm², $\sigma_y = \tau_{xy}$ = 40 N/mm². What is the radius of the Mohr circle for stress drawn with a scale of: 1 cm = 10 N/mm²?** **[IES-2005]**

(a) 3 cm (b) 4 cm (c) 5 cm (d) 6 cm

IES-27. Ans. (c) Radius of the Mohr circle

$$=\left[\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}{}^2}\right]/10 \quad =\left[\sqrt{\left(\frac{100-40}{2}\right)^2+40^2}\right]/10 \;=50/10=5\text{cm}$$

IES-28. Consider a two dimensional state of stress given for an element as shown in the diagram given below: **[IES-2004]**



What are the coordinates of the centre of Mohr's circle?

(a) (0, 0) (b) (100, 200) (c) (200, 100) (d) (50, 0)

IES-28. Ans. (d) Centre of Mohr's circle is $\left(\dfrac{\sigma_x+\sigma_y}{2},0\right)=\left(\dfrac{200-100}{2},0\right)=(50,0)$

IES-29. Two-dimensional state of stress at a point in a plane stressed element is represented by a Mohr circle of zero radius. Then both principal stresses

(a) Are equal to zero **[IES-2003]**
(b) Are equal to zero and shear stress is also equal to zero
(c) Are of equal magnitude but of opposite sign
(d) Are of equal magnitude and of same sign

IES-29. Ans. (d)

IES-30. Assertion (A): Mohr's circle of stress can be related to Mohr's circle of strain by some constant of proportionality. **[IES-2002]**
Reason (R): The relationship is a function of yield stress of the material.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-30. Ans. (c)

IES-31. When two mutually perpendicular principal stresses are unequal but like, the maximum shear stress is represented by **[IES-1994]**

(a) The diameter of the Mohr's circle
(b) Half the diameter of the Mohr's circle
(c) One-third the diameter of the Mohr's circle
(d) One-fourth the diameter of the Mohr's circle

IES-31. Ans. (b)

IES-32. State of stress in a plane element is shown in figure I. Which one of the following figures-II is the correct sketch of Mohr's circle of the state of stress? **[IES-1993, 1996]**

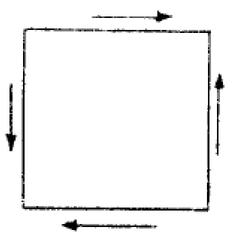


Figure-I

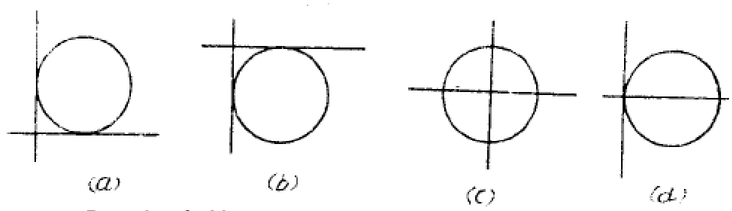


Figure-II

IES-32. Ans. (c)

Strain

IES-33. A point in a two dimensional state of strain is subjected to pure shearing strain of magnitude γ_{xy} radians. Which one of the following is the maximum principal strain? [IES-2008]

(a) γ_{xy} (b) $\gamma_{xy}/\sqrt{2}$ (c) $\gamma_{xy}/2$ (d) $2\gamma_{xy}$

IES-33. Ans. (c)

IES-34. Assertion (A): A plane state of stress does not necessarily result into a plane state of strain as well. [IES-1996]

Reason (R): Normal stresses acting along X and Y directions will also result into normal strain along the Z-direction.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-34. Ans. (a)

Principal strains

IES-35. Principal strains at a point are 100×10^{-6} and -200×10^{-6}. What is the maximum shear strain at the point? [IES-2006]

(a) 300×10^{-6} (b) 200×10^{-6} (c) 150×10^{-6} (d) 100×10^{-6}

IES-35. Ans. (a) $\gamma_{max} = \varepsilon_1 - \varepsilon_2 = 100 - (-200)\times10^{-6} = 300\times10^{-6}$

don't confuse with Maximum Shear stress $(\tau_{max}) = \dfrac{\sigma_1 - \sigma_2}{2}$

in strain $\dfrac{\gamma_{xy}}{2} = \dfrac{\varepsilon_1 - \varepsilon_2}{2}$ and $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2}$ that is the difference.

IES-36. The principal strains at a point in a body, under biaxial state of stress, are 1000×10^{-6} and -600×10^{-6}. What is the maximum shear strain at that point? [IES-2009]

(a) 200×10^{-6} (b) 800×10^{-6} (c) 1000×10^{-6} (d) 1600×10^{-6}

IES-36. Ans. (d)

$$\frac{\epsilon_x - \epsilon_y}{2} = \frac{\phi_{xy}}{2} \quad \Rightarrow \quad \phi_{xy} = \epsilon_x - \epsilon_y = 1000\times10^{-6} - \left(-600\times10^{-6}\right) = 1600\times10^{-6}$$

IES-37. The number of strain readings (using strain gauges) needed on a plane surface to determine the principal strains and their directions is: [IES-1994]

(a) 1 (b) 2 (c) 3 (d) 4

IES-37. Ans. (c) Three strain gauges are needed on a plane surface to determine the principal strains and their directions.

Principal strain induced by principal stress

IES-38. The principal stresses at a point in two dimensional stress system are σ_1 and σ_2 and corresponding principal strains are ε_1 and ε_2. If E and ν denote Young's modulus and Poisson's ratio, respectively, then which one of the following is correct? [IES-2008]

(a) $\sigma_1 = E\varepsilon_1$ (b) $\sigma_1 = \dfrac{E}{1-\nu^2}\left[\varepsilon_1 + \nu\varepsilon_2\right]$

(c) $\sigma_1 = \dfrac{E}{1-\nu^2}\left[\varepsilon_1 - \nu\varepsilon_2\right]$ (d) $\sigma_1 = E\left[\varepsilon_1 - \nu\varepsilon_2\right]$

IES-38. Ans. (b) $\varepsilon_1 = \dfrac{\sigma_1}{E} - \mu\dfrac{\sigma_2}{E}$ *and* $\varepsilon_2 = \dfrac{\sigma_2}{E} - \mu\dfrac{\sigma_1}{E}$ From these two equation eliminate σ_2.

IES-39. Assertion (A): Mohr's construction is possible for stresses, strains and area moment of inertia. [IES-2009]
Reason (R): Mohr's circle represents the transformation of second-order tensor.
(a) Both A and R are individually true and R is the correct explanation of A.
(b) Both A and R are individually true but R is **NOT** the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true.

IES-39. Ans. (a)

Previous 20-Years IAS Questions

Stresses due to Pure Shear

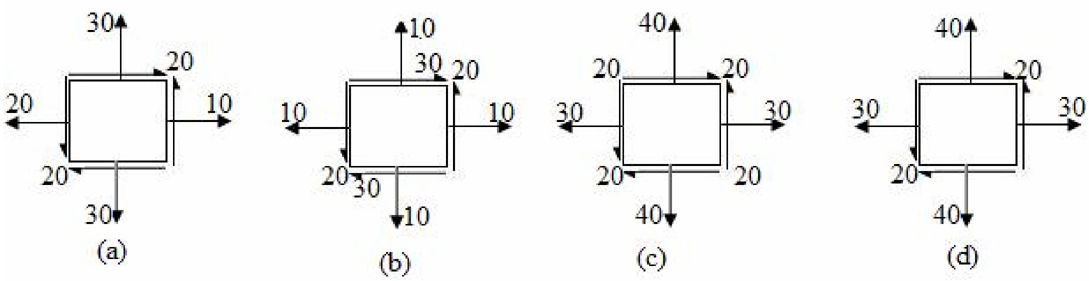
IAS-1. On a plane, resultant stress is inclined at an angle of 45° to the plane. If the normal stress is 100 N /mm², the shear stress on the plane is: [IAS-2003]
(a) 71.5 N/mm² (b) 100 N/mm² (c) 86.6 N/mm² (d) 120.8 N/mm²

IAS-1. Ans. (b) We know $\sigma_n = \sigma\cos^2\theta$ and $\tau = \sigma\sin\theta\cos\theta$

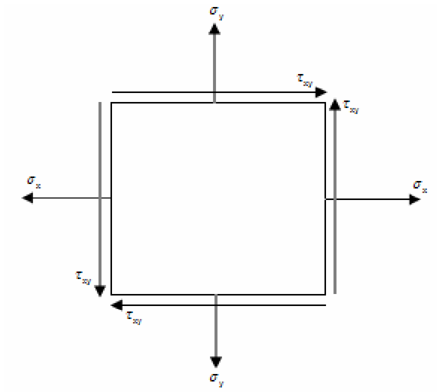
$$100 = \sigma\cos^2 45 \quad \text{or} \quad \sigma = 200$$

$$\tau = 200\sin 45\cos 45 = 100$$

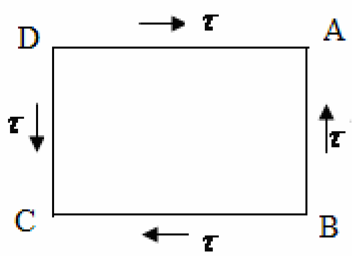
IAS-2. Biaxial stress system is correctly shown in [IAS-1999]



IAS-2. Ans. (c)



IAS-3. The complementary shear stresses of intensity τ are induced at a point in the material, as shown in the figure. Which one of the following is the correct set of orientations of principal planes with respect to AB?
(a) 30° and 120° (b) 45° and 135°
(c) 60° and 150° (d) 75° and 165°



[IAS-1998]

IAS-3. Ans. (b) It is a case of pure shear so principal planes will be along the diagonal.

IAS-4. A uniform bar lying in the *x*-direction is subjected to pure bending. Which one of the following tensors represents the strain variations when bending moment is about the z-axis (p, q and r constants)? [IAS-2001]

(a) $\begin{pmatrix} py & 0 & 0 \\ 0 & qy & 0 \\ 0 & 0 & ry \end{pmatrix}$ (b) $\begin{pmatrix} py & 0 & 0 \\ 0 & qy & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} py & 0 & 0 \\ 0 & py & 0 \\ 0 & 0 & py \end{pmatrix}$ (d) $\begin{pmatrix} py & 0 & 0 \\ 0 & qy & 0 \\ 0 & 0 & qy \end{pmatrix}$

IAS-4. Ans. (d) Stress in *x* direction = σ_x

Therefore $\varepsilon_x = \frac{\sigma_x}{E}, \quad \varepsilon_y = -\mu\frac{\sigma_x}{E}, \quad \varepsilon_z = -\mu\frac{\sigma_x}{E}$

IAS-5. Assuming E = 160 GPa and G = 100 GPa for a material, a strain tensor is given as: [IAS-2001]

$$\begin{pmatrix} 0.002 & 0.004 & 0.006 \\ 0.004 & 0.003 & 0 \\ 0.006 & 0 & 0 \end{pmatrix}$$

The shear stress, τ_{xy} is:

(a) 400 MPa (b) 500 MPa (c) 800 MPa (d) 1000 MPa

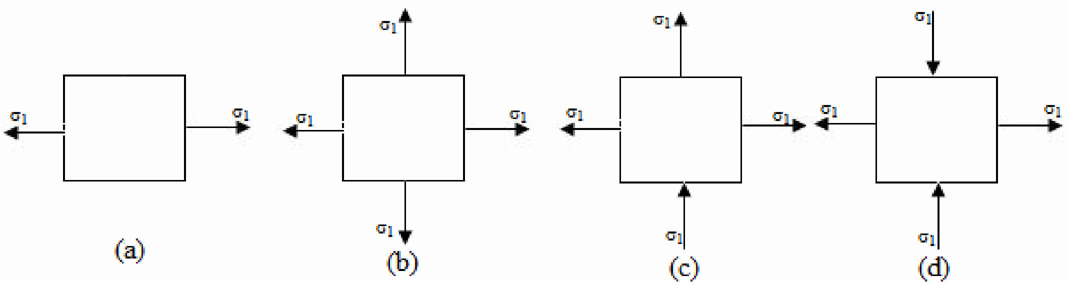
IAS-5. Ans. (c)

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \text{ and } \varepsilon_{xy} = \frac{\gamma_{xy}}{2}$$

$$\tau_{xy} = G\,\gamma_{xy} = 100 \times 10^3 \times (0.004 \times 2)\,\text{MPa} = 800\text{MPa}$$

Principal Stress and Principal Plane

IAS-6. A material element subjected to a plane state of stress such that the maximum shear stress is equal to the maximum tensile stress, would correspond to [IAS-1998]



IAS-6. Ans. (d) $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_1 - (-\sigma_1)}{2} = \sigma_1$

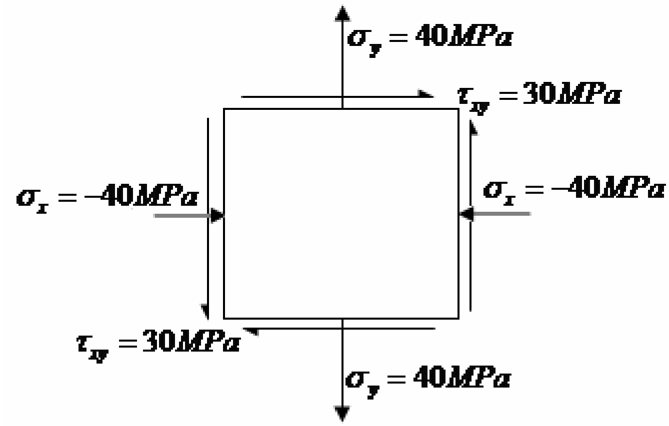
IAS-7. A solid circular shaft is subjected to a maximum shearing stress of 140 MPs. The magnitude of the maximum normal stress developed in the shaft is: [IAS-1995]

(a) 140 MPa (b) 80 MPa (c) 70 MPa (d) 60 MPa

IAS-7. Ans. (a) $\tau_{\max} = \dfrac{\sigma_1 - \sigma_2}{2}$ Maximum normal stress will developed if $\sigma_1 = -\sigma_2 = \sigma$

IAS-8. **The state of stress at a point in a loaded member is shown in the figure. The magnitude of maximum shear stress is [1MPa = 10 kg/cm²] [IAS 1994]**

(a) 10 MPa (b) 30 MPa (c) 50 MPa (d) 100MPa



IAS-8. Ans. (c) $\tau_{\max} = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \sqrt{\left(\dfrac{-40-40}{2}\right)^2 + 30^2}$ = 50 MPa

IAS-9. **A horizontal beam under bending has a maximum bending stress of 100 MPa and a maximum shear stress of 20 MPa. What is the maximum principal stress in the beam? [IAS-2004]**

(a) 20 (b) 50 (c) 50 + $\sqrt{2900}$ (d) 100

IAS-9. Ans. (c) σ_b=100MP$_a$ τ =20 mP$_a$

$$\sigma_{1,2} = \frac{\sigma_b}{2} + \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2}$$

$$\sigma_{1,2} = \frac{\sigma_b}{2} + \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} = \frac{100}{2} + \sqrt{\left(\frac{100}{2}\right)^2 + 20^2} = \left(50 + \sqrt{2900}\right)\text{MPa}$$

IAS-10. **When the two principal stresses are equal and like: the resultant stress on any plane is: [IAS-2002]**

(a) Equal to the principal stress (b) Zero
(c) One half the principal stress (d) One third of the principal stress

IAS-10. Ans. (a) $\sigma_n = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta$

[We may consider this as $\tau_{xy} = 0$] $\sigma_x = \sigma_y = \sigma(say)$ So $\sigma_n = \sigma$ for any plane

IAS-11. **Assertion (A): When an isotropic, linearly elastic material is loaded biaxially, the directions of principal stressed are different from those of principal strains. [IAS-2001]**

Reason (R): For an isotropic, linearly elastic material the Hooke's law gives only two independent material properties.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-11. Ans. (d) They are same.

IAS-12. Principal stress at a point in a stressed solid are 400 MPa and 300 MPa respectively. The normal stresses on planes inclined at 45° to the principal planes will be: [IAS-2000]

(a) 200 MPa and 500 MPa (b) 350 MPa on both planes
(c) 100MPaand6ooMPa (d) 150 MPa and 550 MPa

IAS-12. Ans. (b)

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta = \frac{400+300}{2} + \frac{400-300}{2}\cos 2\times 45^o = 350 MPa$$

IAS-13. The principal stresses at a point in an elastic material are 60N/mm² tensile, 20 N/mm² tensile and 50 N/mm² compressive. If the material properties are: μ = 0.35 and E = 105 Nmm², then the volumetric strain of the material is: [IAS-1997]

(a) 9×10^{-5} (b) 3×10^{-4} (c) 10.5×10^{-5} (d) 21×10^{-5}

IAS-13. Ans. (a)

$$\in_x = \frac{\sigma_x}{E} - \mu\left(\frac{\sigma_y}{E} + \frac{\sigma_z}{E}\right), \quad \in_y = \frac{\sigma_y}{E} - \mu\left(\frac{\sigma_z}{E} + \frac{\sigma_x}{E}\right) \text{ and } \in_z = \frac{\sigma_z}{E} - \mu\left(\frac{\sigma_x}{E} + \frac{\sigma_y}{E}\right)$$

$$\in_v = \in_x + \in_y + \in_z = \frac{\sigma_x + \sigma_y + \sigma_z}{E} - \frac{2\mu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

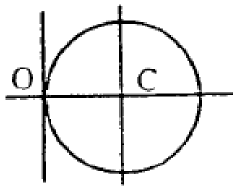
$$= (1-2\mu)\left(\frac{\sigma_x + \sigma_y + \sigma_z}{E}\right) = \left(\frac{60+20-50}{10^5}\right)(1-2\times 0.35) = 9\times 10^{-5}$$

Mohr's circle

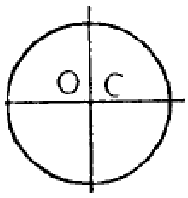
IAS-14. Match List-I (Mohr's Circles of stress) with List-II (Types of Loading) and select the correct answer using the codes given below the lists: [IAS-2004]

List-I (Mohr's Circles of Stress)

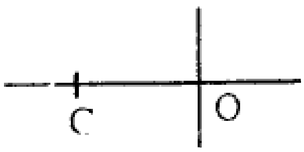
A.



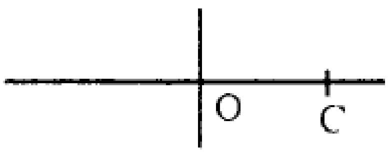
B.



C.



D.



List-II (Types of Loading)

1. A shaft compressed all round by a hub
2. Bending moment applied at the free end of a cantilever
3. Shaft under torsion
4. Thin cylinder under pressure
5. Thin spherical shell under internal pressure

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	5	4	3	2	(b)	2	4	1	3
(c)	4	3	2	5	(d)	2	3	1	5

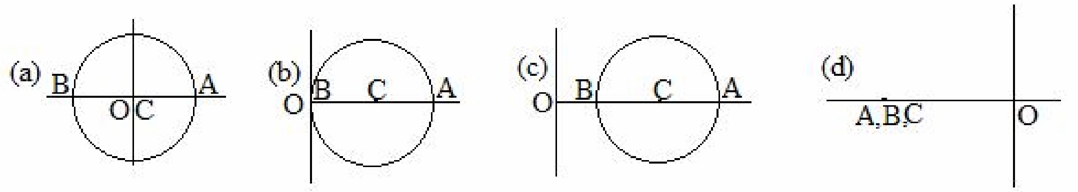
IAS-14. Ans. (d)

IAS-15. The resultant stress on a certain plane makes an angle of 20° with the normal to the plane. On the plane perpendicular to the above plane, the resultant stress makes an angle of θ with the normal. The value of θ can be: [IAS-2001]

(a) 0° or 20°
(b) Any value other than 0° or 90°
(c) Any value between 0° and 20°
(d) 20° only

IAS-15. Ans. (b)

IAS-16. The correct Mohr's stress-circle drawn for a point in a solid shaft compressed by a shrunk fit hub is as (O-Origin and C-Centre of circle; OA = σ_1 and OB = σ_2) [IAS-2001]



IAS-16. Ans. (d)

IAS-17. A Mohr's stress circle is drawn for a body subjected to tensile stress f_x and f_y in two mutually perpendicular directions such that $f_x > f_y$. Which one of the following statements in this regard is NOT correct? [IAS-2000]

(a) Normal stress on a plane at 45° to f_x is equal to $\dfrac{f_x + f_y}{2}$

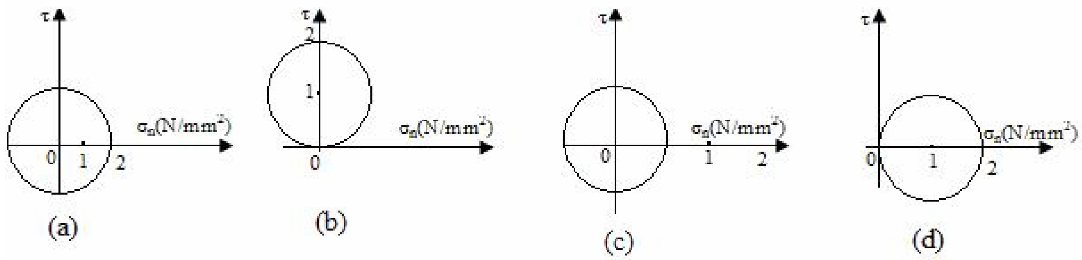
(b) Shear stress on a plane at 450 to f_x is equal to $\dfrac{f_x - f_y}{2}$

(c) Maximum normal stress is equal to f_x.

(d) Maximum shear stress is equal to $\dfrac{f_x + f_y}{2}$

IAS-17. Ans. (d) Maximum shear stress is $\dfrac{f_x - f_y}{2}$

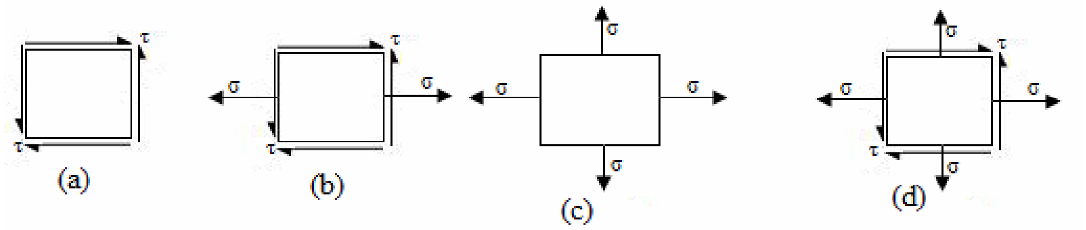
IAS-18. For the given stress condition σ_x=2 N/mm², σ_x=0 and $\tau_{xy} = 0$, the correct Mohr's circle is: [IAS-1999]



IAS-18. Ans. (d) Centre$\left(\dfrac{\sigma_x + \sigma_y}{2}, 0\right) = \left(\dfrac{2+0}{2}, 0\right) = (1, 0)$

$$\text{radius} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_x^2} = \sqrt{\left(\frac{2-0}{2}\right)^2 + 0} = 1$$

IAS-19. For which one of the following two-dimensional states of stress will the Mohr's stress circle degenerate into a point? [IAS-1996]



IAS-19. Ans. (c) Mohr's circle will be a point.

Radius of the Mohr's circle = $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ $\therefore \tau_{xy} = 0$ and $\sigma_x = \sigma_y = \sigma$

Principal strains

IAS-20. In an axi-symmetric plane strain problem, let u be the radial displacement at r. Then the strain components $\varepsilon_r, \varepsilon_\theta, \Upsilon_{e\theta}$ are given by [IAS-1995]

(a) $\varepsilon_r = \frac{u}{r}, \varepsilon_\theta = \frac{\partial u}{\partial r}, \Upsilon_{r\theta} = \frac{\partial^2 u}{\partial r \partial \theta}$

(b) $\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\theta = \frac{u}{r}, \Upsilon_{r\theta} = o$

(c) $\varepsilon_r = \frac{u}{r}, \varepsilon_\theta = \frac{\partial u}{\partial r}, \Upsilon_{r\theta} = 0$

(d) $\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\theta = \frac{\partial u}{\partial \theta}, \Upsilon_{r\theta} = \frac{\partial^2 u}{\partial r \partial \theta}$

IAS-20. Ans. (b)

IAS-21. Assertion (A): Uniaxial stress normally gives rise to triaxial strain.
Reason (R): Magnitude of strains in the perpendicular directions of applied stress is smaller than that in the direction of applied stress. [IAS-2004]

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-21. Ans. (b)

IAS-22. Assertion (A): A plane state of stress will, in general, not result in a plane state of strain. [IAS-2002]
Reason (R): A thin plane lamina stretched in its own plane will result in a state of plane strain.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-22. Ans. (c) R is false. Stress in one plane always induce a lateral strain with its orthogonal plane.

Previous Conventional Questions with Answers

Conventional Question IES-1999

***Question*:** **What are principal in planes?**

***Answer*:** The planes which pass through the point in such a manner that the resultant stress across them is totally a normal stress are known as principal planes. No shear stress exists at the principal planes.

Conventional Question IES-2009

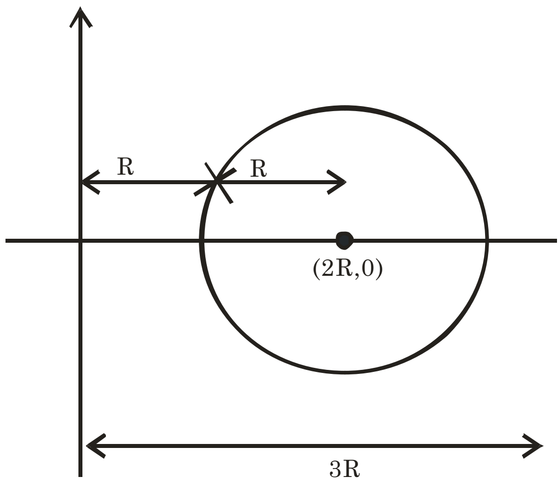
Q. **The Mohr's circle for a plane stress is a circle of radius R with its origin at + 2R on σ axis. Sketch the Mohr's circle and determine σ_{max}, σ_{min}, σ_{av}, $\left(\tau_{xy}\right)_{max}$ for this situation.** **[2 Marks]**

Ans. Here $\sigma_{max} = 3R$

$\sigma_{min} = R$

$$\sigma_{\sigma v} = \frac{3R + R}{2} = 2R$$

$$\text{and } \tau_{xy} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{3R - R}{2} = R$$



Conventional Question IES-1999

***Question*:** **Direct tensile stresses of 120 MPa and 70 MPa act on a body on mutually perpendicular planes. What is the magnitude of shearing stress that can be applied so that the major principal stress at the point does not exceed 135 MPa? Determine the value of minor principal stress and the maximum shear stress.**

***Answer*:** Let shearing stress is 'τ' MPa.

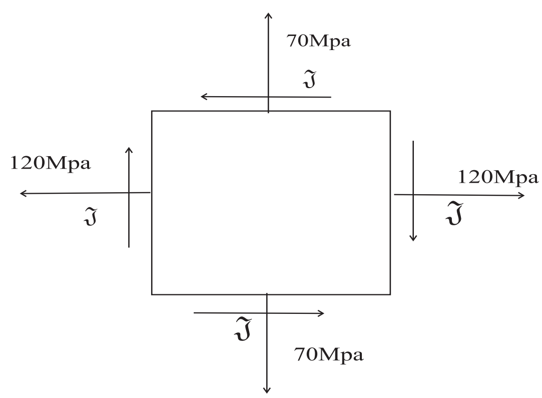
The principal stresses are

$$\sigma_{1,2} = \frac{120 + 70}{2} \pm \sqrt{\left(\frac{120 - 70}{2}\right)^2 + \tau^2}$$

Major principal stress is

$$\sigma_1 = \frac{120 + 70}{2} + \sqrt{\left(\frac{120 - 70}{2}\right)^2 + \tau^2}$$

$$= 135 (\text{Given}) \ or, \tau = 31.2 MPa.$$



Minor principal stress is

$$\sigma_2 = \frac{120+70}{2} - \sqrt{\left(\frac{120-70}{2}\right)^2 + 31.2^2} = 55\,\text{MPa}$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{135-55}{2} = 40\,\text{MPa}$$

Conventional Question IES-2009

Q. **The state of stress at a point in a loaded machine member is given by the principle stresses.** **[2 Marks]**

$\boldsymbol{\sigma_1 = 600}$ **MPa,** $\boldsymbol{\sigma_2 = 0}$ and $\boldsymbol{\sigma_3 = -600}$ **MPa**.

(i) What is the magnitude of the maximum shear stress?

(ii) What is the inclination of the plane on which the maximum shear stress acts with respect to the plane on which the maximum principle stress $\boldsymbol{\sigma_1}$ acts?

Ans. **(i)** Maximum shear stress,

$$\tau = \frac{\sigma_1 - \sigma_3}{2} = \frac{600 - (-600)}{2}$$

$$= 600\ \text{MPa}$$

(ii) At $\theta = 45^\circ$ max. shear stress occurs with σ_1 plane. Since σ_1 and σ_3 are principle stress does not contains shear stress. Hence max. shear stress is at 45° with principle plane.

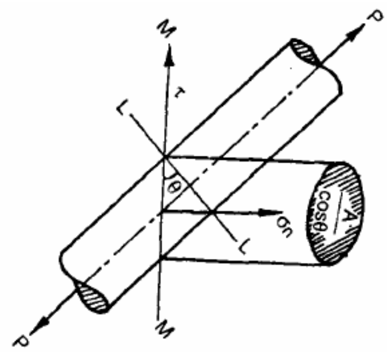
Conventional Question IES-2008

***Question*:** **A prismatic bar in compression has a cross- sectional area A = 900 mm² and carries an axial load P = 90 kN. What are the stresses acts on**

(i) A plane transverse to the loading axis;

(ii) A plane at $\theta = 60^\circ$ to the loading axis?

***Answer*:** (i) From figure it is clear A plane transverse to loading axis, $\theta = 0^\circ$



$$\therefore \sigma_n = \frac{P}{A}\cos^2\theta = \frac{90000}{900} N/mm^2$$

$$= 100\, N/mm^2$$

$$\text{and } \tau = \frac{P}{2A}\text{Sin}2\theta = \frac{90000}{2\times 900}\times \sin\theta = 0$$

(iii) A plane at 60° to loading axis,

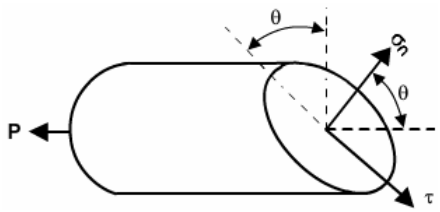
$\theta = 60° - 30° = 30°$

$$\sigma_n = \frac{P}{A}\cos^2\theta = \frac{90000}{900}\times\cos^2 30$$

$$= 75 N/mm^2$$

$$\tau = \frac{P}{2A}\sin 2\theta = \frac{90000}{2\times 900}\sin 2\times 60^o$$

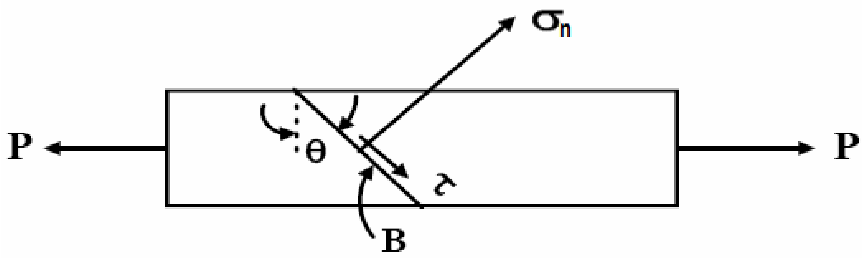
$$= 43.3 N/mm^2$$



Conventional Question IES-2001

***Question*:** **A tension member with a cross-sectional area of 30 mm² resists a load of 80 kN, Calculate the normal and shear stresses on the plane of maximum shear stress.**

***Answer*:** $\sigma_n = \dfrac{P}{A}\cos^2\theta$ $\qquad$ $\tau = \dfrac{P}{2A}\sin 2\theta$



For maximum shear stress $\sin 2\theta = 1$, or, $\theta = 45^o$

$$(\sigma_n) = \frac{80\times10^3}{30}\times\cos^2 45 = 1333MPa \text{ and } \tau_{max} = \frac{P}{2A} = \frac{80\times10^3}{30\times2} = 1333MPa$$

Conventional Question IES-2007

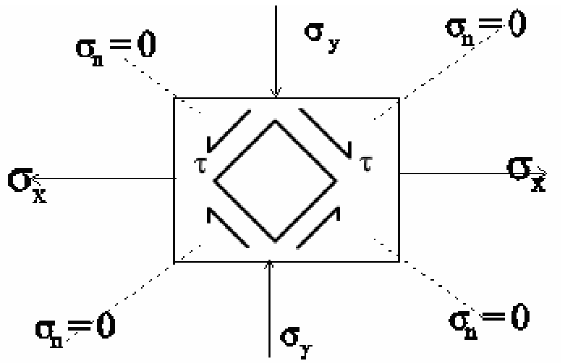
***Question*:** **At a point in a loaded structure, a pure shear stress state τ = $\pm$400 MPa prevails on two given planes at right angles.**

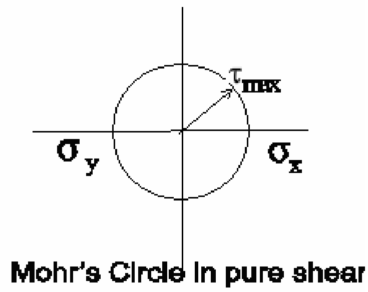
(i) What would be the state of stress across the planes of an element taken at +45° to the given planes?

(ii) What are the magnitudes of these stresses?

***Answer*:** (i) For pure shear

$$\sigma_x = -\sigma_y; \qquad \tau_{max} = \pm\sigma_x = \pm 400 MPa$$





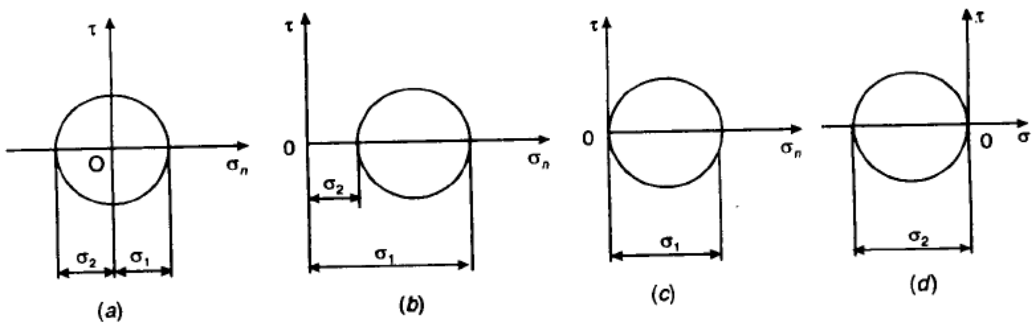
(ii) Magnitude of these stresses

$$\sigma_n = \tau_{xy} Sin 2\theta = \tau_{xy} Sin 90^o = \tau_{xy} = 400 MPa \text{ and } \tau = (-\tau_{xy}\cos 2\theta) = 0$$

Conventional Question IAS-1997

***Question*:** **Draw Mohr's circle for a 2-dimensional stress field subjected to (a) Pure shear (b) Pure biaxial tension (c) Pure uniaxial tension and (d) Pure uniaxial compression**

***Answer*:** Mohr's circles for 2-dimensional stress field subjected to pure shear, pure biaxial tension, pure uniaxial compression and pure uniaxial tension are shown in figure below:

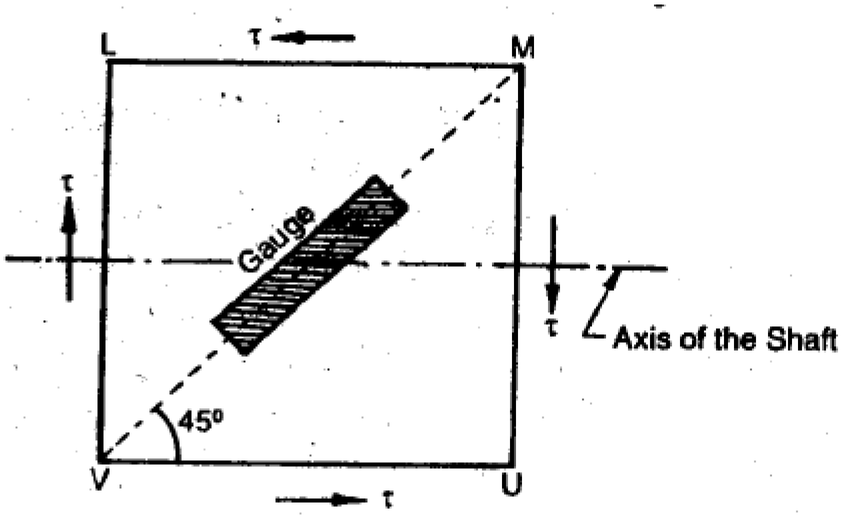


Conventional Question IES-2003

***Question*:** **A Solid phosphor bronze shaft 60 mm in diameter is rotating at 800 rpm and transmitting power. It is subjected torsion only. An electrical resistance**

strain gauge mounted on the surface of the shaft with its axis at 45° to the shaft axis, gives the strain reading as 3.98×10^{-4}. If the modulus of elasticity for bronze is 105 GN/m^2 and Poisson's ratio is 0.3, find the power being transmitted by the shaft. Bending effect may be neglected.

Answer:



Let us assume maximum shear stress on the cross-sectional plane MU is τ . Then

Principal stress along, VM = $-\frac{1}{2}\sqrt{4\tau^2} = -\tau$ (compressive)

Principal stress along, LU = $\frac{1}{2}\sqrt{4\tau^2} = \tau$(tensile)

Thus magntude of the compressive strain along VM is

$$=\frac{\tau}{E}(1+\mu) = 3.98\times10^{-4}$$

$$or\ \tau = \frac{3.98\times10^{-4}\times\left(105\times10^{9}\right)}{(1+0.3)} = 32.15\,MPa$$

$\therefore$ Torque being transmitted (T) = $\tau\times\frac{\pi}{16}\times d^3$

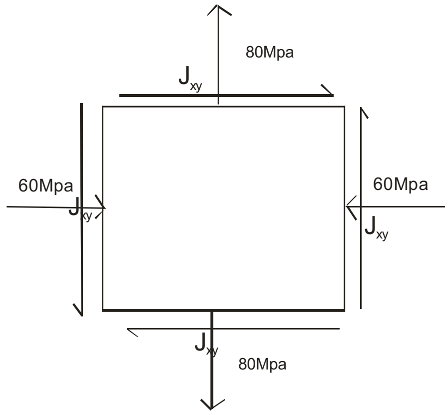
$$=\left(32.15\times10^{6}\right)\times\frac{\pi}{16}\times0.06^3 = 1363.5 \text{ Nm}$$

$\therefore$ Power being transmitted, P $=T.\omega=T.\left(\frac{2\pi N}{60}\right) = 1363.5\times\left(\frac{2\pi\times800}{60}\right)W = 114.23\,kW$

Conventional Question IES-2002

Question: **The magnitude of normal stress on two mutually perpendicular planes, at a point in an elastic body are 60 MPa (compressive) and 80 MPa (tensile) respectively. Find the magnitudes of shearing stresses on these planes if the magnitude of one of the principal stresses is 100 MPa (tensile). Find also the magnitude of the other principal stress at this point.**

Answer: Above figure shows stress condition assuming shear stress is ' τ_{xy}'



Principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$or, \sigma_{1,2} = \frac{-60+80}{2} \pm \sqrt{\left(\frac{-60-80}{2}\right)^2 + \tau_{xy}^2}$$

$$or, \sigma_{1,2} = \frac{-60+80}{2} \pm \sqrt{\left(\frac{-60-80}{2}\right)^2 + \tau_{xy}^2}$$

To make principal stress 100 MPa we have to consider '+' .

$$\therefore \sigma_1 = 100\,\text{MPa} = 10 + \sqrt{70^2 + \tau_{xy}^2}\,; \quad \text{or, } \tau_{xy} = 56.57\,\text{MPa}$$

Therefore other principal stress will be

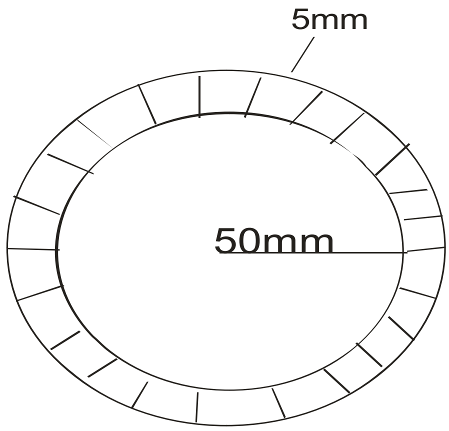
$$\sigma_2 = \frac{-60+80}{2} - \sqrt{\left(\frac{-60-80}{2}\right)^2 + (56.57)^2}$$

i.e. 80 MPa(compressive)

Conventional Question IES-2001

***Question*: A steel tube of inner diameter 100 mm and wall thickness 5 mm is subjected to a torsional moment of 1000 Nm. Calculate the principal stresses and orientations of the principal planes on the outer surface of the tube.**

Answer: Polar moment of Inertia (J)= $\frac{\pi}{32}\left[(0.110)^4 - (0.100)^4\right] = 4.56 \times 10^{-6} m^4$



$$\text{Now } \frac{T}{J} = \frac{\tau}{R} \text{ or } J = \frac{T.R}{J} = \frac{1000 \times (0.055)}{4.56 \times 10^{-6}}$$

$$= 12.07\,\text{MPa}$$

$$\text{Now}, \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \infty,$$

$$\text{gives } \theta_p = 45^0 \text{ or } 135^0$$

$$\therefore \sigma_1 = \tau_{xy} Sin 2\theta = 12.07 \times \sin 90^0$$

$$= 12.07\,MPa$$

$$and\ \sigma_2 = 12.07 \sin 270^0$$

$$= -12.07 MPa$$

Conventional Question IES-2000

***Question*: At a point in a two dimensional stress system the normal stresses on two mutually perpendicular planes are σ_x and σ_y and the shear stress is τ_{xy}. At what value of shear stress, one of the principal stresses will become zero?**

Answer: Two principal stressdes are

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Considering (-)ive sign it may be zero

$$\therefore\left(\frac{\sigma_x+\sigma_y}{2}\right)=\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\qquad \text{or,}\left(\frac{\sigma_x+\sigma_y}{2}\right)^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2$$

$$\text{or,}\tau_{xy}^2=\left(\frac{\sigma_x+\sigma_y}{2}\right)^2-\left(\frac{\sigma_x-\sigma_y}{2}\right)^2\qquad \text{or,}\tau_{xy}^2=\sigma_x\sigma_y\qquad \text{or,}\tau_{xy}=\pm\sqrt{\sigma_x\sigma_y}$$

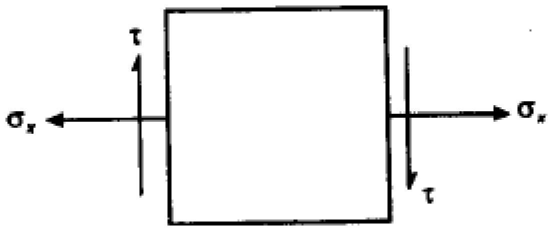
Conventional Question IES-1996

Question: **A solid shaft of diameter 30 mm is fixed at one end. It is subject to a tensile force of 10 kN and a torque of 60 Nm. At a point on the surface of the shaft, determine the principle stresses and the maximum shear stress.**

Answer: Given: D = 30 mm = 0.03 m; P = 10 kN; T= 60 Nm

Principal stresses (σ_1,σ_2) and maximum shear stress (τ_{max}):

$$\text{Tensile stress } \sigma_t=\sigma_x=\frac{10\times10^3}{\frac{\pi}{4}\times0.03^2}=14.15\times10^6\ \text{N/m}^2 \text{ or } 14.15\ \text{MN/m}^2$$



$$\text{As per torsion equation,}\ \frac{T}{J}=\frac{\tau}{R}$$

$$\therefore \text{Shear stress,}\quad \tau=\frac{TR}{J}=\frac{TR}{\frac{\pi}{32}D^4}=\frac{60\times0.015}{\frac{\pi}{32}\times(0.03)^4}=11.32\times10^6\text{N/m}^2$$

or $11.32\ \text{MN/m}^2$

The principal stresses are calculated by using the relations:

$$\sigma_{1,2}=\left(\frac{\sigma_x+\sigma_y}{2}\right)\pm\sqrt{\left[\left(\frac{\sigma_x-\sigma_y}{2}\right)^2\right]+\tau_{xy}^2}$$

Here $\sigma_x=14.15\text{MN/m}^2,\sigma_y=0;\tau_{xy}=\tau=11.32\quad \text{MN/m}^2$

$$\therefore\quad \sigma_{1,2}=\frac{14.15}{2}\pm\sqrt{\left(\frac{14.15}{2}\right)^2+(11.32)^2}$$

$$=7.07\pm13.35=20.425\ \text{MN/m}^2,-6.275\text{MN/m}^2.$$

Hence, major principal stress, $\sigma_1=20.425\ \text{MN/m}^2\ (\text{tensile})$

Minor principal stress, $\sigma_2=6.275\text{MN/m}^2\ (\text{compressive})$

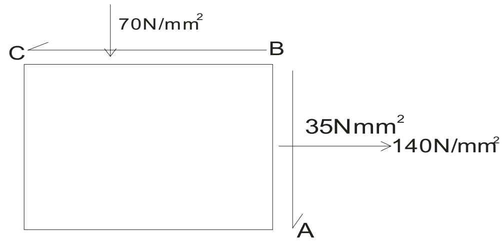
$$\text{Maximum shear stress,}\tau_{max}=\frac{\sigma_1-\sigma_2}{2}=\frac{24.425-(-6.275)}{2}=13.35\text{mm/m}^2$$

Conventional Question IES-2000

Question: **Two planes AB and BC which are at right angles are acted upon by tensile stress of 140 N/mm² and a compressive stress of 70 N/mm² respectively and also by stress 35 N/mm². Determine the principal stresses and principal planes. Find also the maximum shear stress and planes on which they act.**

Sketch the Mohr circle and mark the relevant data.

Answer: Given



σ_x=140 MPa (tensile)

σ_y=-70 MPa (compressive)

$\tau_{xy} = 35$MPa

Principal stresses; σ_1, σ_2;

We know that, $\sigma_{1,2} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$$= \frac{140 - 70}{2} \pm \sqrt{\left(\frac{140 + 70}{2}\right)^2 + 35^2} = 35 \pm 110.7$$

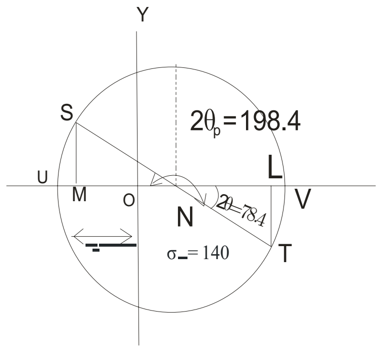
Therefore σ_1=145.7 MPa and $\sigma_2 = -75.7$MPa

Position of Principal planes θ_1, θ_2

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times 35}{140 + 70} = 0.3333$$

Maximum shear stress, $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{145 + 75.7}{2} = 110.7MPa$

<u>Mohr cirle:</u>



OL=$\sigma_x = 140MPa$

$OM = \sigma_y = -70MPa$

$SM = LT = \tau_{xy} = 35MPa$

Joining ST that cuts at 'N'

SN=NT=radius of Mohr circle =110.7 MPa

OV=$\sigma_1 = 145.7MPa$

$OV = \sigma_2 = -75.7MPa$

Conventional Question IES-2010

Q6. The data obtained from a rectangular strain gauge rosette attached to a stressed steel member are $\in_0 = -220\times10^{-6}$, $\in_{45}^0 = 120\times100^{-6}$, and $\in_{90} = 220\times10^{-6}$. Given that the value of E = $2\times10^5\ \text{N/mm}^2$ and Poisson's Ratio $\mu = 0.3$, calculate the values of principal stresses acting at the point and their directions. [10 Marks]

Ans. A rectangular strain gauge rosette strain

$\in_0 = -220\times10^{-6}$ $\in_{45^0} = 120\times10^{-6}$ $\in_{90} = 220\times10^{-6}$

$E = 2\times10^{11}\ \text{N/m}^2$ poisson ratio $\mu = 0.3$

Find out principal stress and their direction.

Let $\mathbf{e_a} = \in_o$ $\mathbf{e_c} = \in_{90}$ **and** $\mathbf{e_b} = \in_{45}$

We know that principal strain are

$$\in_{12} = \frac{\mathbf{e_a} + \mathbf{e_b}}{2} \sqrt{(\mathbf{e_a} - \mathbf{e_b})^2 + (\mathbf{e_b} - \mathbf{e_c})^2}$$

$$\Rightarrow \frac{(-220\times10^{-6} + 120\times10^{-6})}{2} \pm \frac{1}{\sqrt{2}}\sqrt{((-220 - 120) + 10^{-6})^2 + ((120 - 220)10^{-6})^2}$$

$$\Rightarrow -50\times10^{-6} \pm \frac{1}{\sqrt{2}} 354.40\times10^{-6}$$

$$\in_{12} \Rightarrow -50\times10^{-6} \pm 250.6\times10^{-6}$$

$$\in_1 = 2.01\times10^{-4}$$

$$\in_2 = -3.01\times10^{-4}$$

Direction can be find out : -

$$\tan 2\theta_{p\in} = \frac{2e_b - e_a - e_c}{e_c - e_a} = \frac{2\times120\times10^{-6}}{220\times10^{-6} + 220\times10^{-6}}$$

$$\Rightarrow \frac{240}{440} = 0.55$$

$$2\theta_{p\in} = 28.81$$

$$\theta_{p\in} = 14.45^0 \text{ clockwise form principal strain } t_1$$

Principal stress:-

$$\sigma_1 \frac{E(\in_1 + \mu\in_2)}{1-\mu^2} = \frac{2\times10^{11}\left(2+0.3(-3)\times10^{-4}\right)}{1-0.3^2}$$

$$= 241.78\times10^5\ N/m^2$$

$$= -527.47\times10^5\ N/m^2$$

Conventional Question IES-1998

Question: **When using strain-gauge system for stress/force/displacement measurements how are in-built magnification and temperature compensation achieved?**

Answer: In-built magnification and temperature compensation are achieved by

(a) Through use of adjacent arm balancing of Wheat-stone bridge.

(b) By means of self temperature compensation by selected melt-gauge and dual element-gauge.

Conventional Question AMIE-1998

Question: **A cylinder (500 mm internal diameter and 20 mm wall thickness) with closed ends is subjected simultaneously to an internal pressure of 0-60 MPa, bending moment 64000 Nm and torque 16000 Nm. Determine the maximum tensile stress and shearing stress in the wall.**

Answer: Given: d = 500 mm = 0·5 m; t = 20 mm = 0·02 m; p = 0·60 MPa = 0.6 MN/m²;
M = 64000 Nm = 0·064 MNm; T= 16000 Nm = 0·016 MNm.

Maximum tensile stress:

First let us determine the principle stresses σ_1 and σ_2 assuming this as a thin cylinder.

We know, $\sigma_1 = \frac{pd}{2t} = \frac{0.6\times0.5}{2\times0.02} = 7.5 MN/m^2$

and $\sigma_2 = \frac{pd}{4t} = \frac{0.6\times0.5}{4\times0.02} = 3.75 MN/m^2$

Next consider effect of combined bending moment and torque on the walls of the cylinder. Then the principal stresses σ'_1 and σ'_2 are given by

$$\sigma'_1 = \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

and $$\sigma'_2 = \frac{16}{\pi d^3}\left[M - \sqrt{M^2 + T^2}\right]$$

$\therefore$ $$\sigma'_1 = \frac{16}{\pi \times (0.5)^3}\left[0.064 + \sqrt{0.064^2 + 0.016^2}\right] = 5.29 MN/m^2$$

and $$\sigma'_2 = \frac{16}{\pi \times (0.5)^3}\left[0.064 - \sqrt{0.064^2 + 0.016^2}\right] = -0.08 MN/m^2$$

Maximum shearing stress, τ_{max} :

We Know, $\tau_{max} = \dfrac{\sigma_I - \sigma_{II}}{2}$

$$\sigma_{II} = \sigma_2 + \sigma'_2 = 3.75 - 0.08 = 3.67 MN/m^2 \,(\text{tensile})$$

$\therefore$ $$\tau_{max} = \frac{12.79 - 3.67}{2} = 4.56 MN/m^2$$

3. Moment of Inertia and Centroid

Theory at a Glance (for IES, GATE, PSU)

3.1 Centre of gravity

The centre of gravity of a body defined as the point through which the whole weight of a body may be assumed to act.

3.2 Centroid or Centre of area

The centroid or centre of area is defined as the point where the whole area of the figure is assumed to be concentrated.

3.3 Moment of Inertia (MOI)

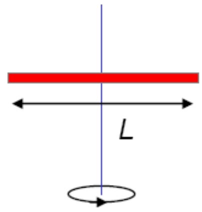
- About any point the product of the force and the perpendicular distance between them is known as moment of a force or first moment of force.
- This first moment is again multiplied by the perpendicular distance between them to obtain second moment of force.
- In the same way if we consider the area of the figure it is called second moment of area or area moment of inertia and if we consider the mass of a body it is called second moment of mass or mass moment of Inertia.
- ***Mass moment of inertia*** is the measure of resistance of the body to rotation and ***forms the basis of dynamics of rigid bodies.***
- ***Area moment of Inertia*** is the measure of resistance to bending and ***forms the basis of strength of materials.***

3.4 Mass moment of Inertia (MOI)

$$I = \sum_i m_i r_i^2$$

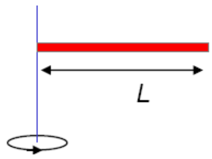
- Notice that the moment of inertia 'I' depends on the distribution of mass in the system.
- The furthest the mass is from the rotation axis, the bigger the moment of inertia.
- For a given object, the moment of inertia depends on where we choose the rotation axis.
- In rotational dynamics, the moment of inertia 'I' appears in the same way that mass m does in linear dynamics.

- ***Solid disc or cylinder of mass M and radius R,*** about perpendicular axis through its centre, $I = \frac{1}{2}MR^2$
- Solid sphere of mass M and radius R, about an axis through its centre, I = 2/5 M R^2
- Thin ***rod of mass M and length L,*** about a perpendicular axis through its centre.



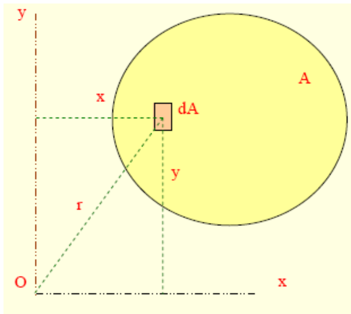
$$I = \frac{1}{12}ML^2$$

- Thin rod of mass M and length L, about a perpendicular axis through its end.



$$I = \frac{1}{3}ML^2$$

3.5 Area Moment of Inertia (MOI) or Second moment of area

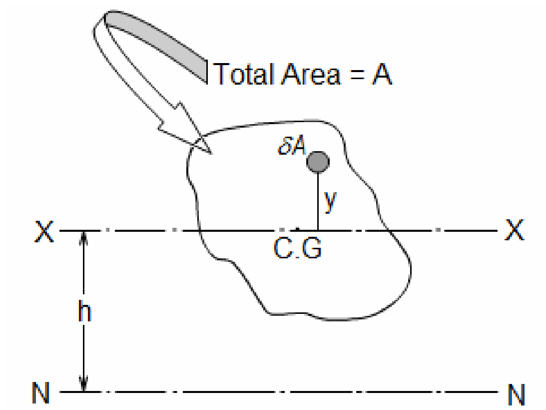


- To find the centroid of an area by the first moment of the area about an axis was determined ($\int x \, dA$)
- Integral of the ***second moment of area*** is called moment of inertia ($\int x^2 dA$)
- Consider the area (A)
- By definition, the moment of inertia of the differential area about the x and y axes are dI_{xx} and dI_{yy}
- $dI_{xx} = y^2 dA$ $\quad$ $\mathbf{I_{xx} = \int y^2 \, dA}$
- $dI_{yy} = x^2 dA$ $\quad$ $\mathbf{I_{yy} = \int x^2 \, dA}$

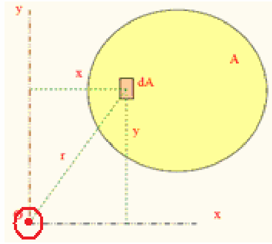
3.6 Parallel axis theorem for an area

The rotational inertia about any axis is the sum of second moment of inertia about a parallel axis through the C.G and total area of the body times square of the distance between the axes.

$$\mathbf{I_{NN} = I_{CG} + Ah^2}$$



3.7 Perpendicular axis theorem for an area



If x, y & z are mutually perpendicular axes as shown, then $I_{zz}(J) = I_{xx} + I_{yy}$

Z-axis is perpendicular to the plane of x – y and vertical to this page as shown in figure.

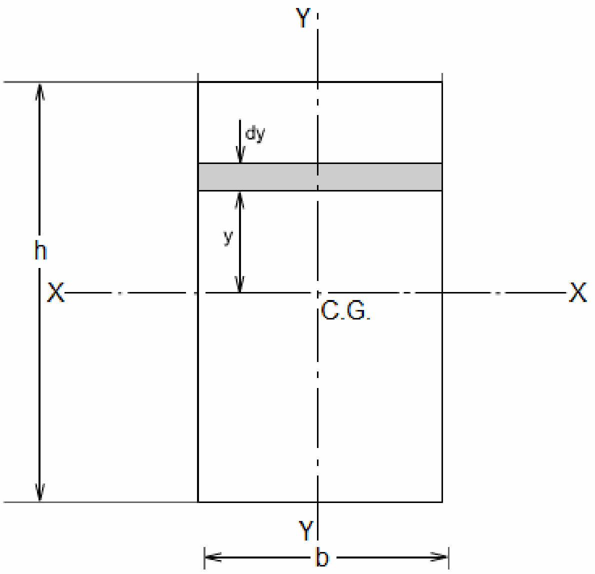
- To find the moment of inertia of the differential area about the pole (point of origin) or z-axis, **(r)** is used. (r) is the perpendicular distance from the pole to dA for the entire area

  $\boldsymbol{J = \int r^2\,dA = \int (x^2 + y^2)dA = I_{xx} + I_{yy}}$ (since $r^2 = x^2 + y^2$)

  Where, J = polar moment of inertia

3.8 Moments of Inertia (area) of some common area

(i) MOI of Rectangular area



Moment of inertia about axis XX which passes through centroid.

Take an element of width 'dy' at a distance y from XX axis.

∴ Area of the element (dA) = b × dy.

and Moment of Inertia of the element about XX axis $= dA \times y^2 = b.y^2.dy$

∴ **Total MOI about XX axis** (Note it is area moment of Inertia)

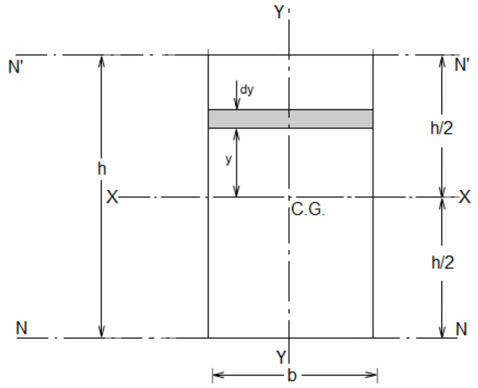
$$I_{xx} = \int_{-h/2}^{+h/2} by^2dy = 2\int_0^{h/2} by^2dy = \frac{bh^3}{12}$$

$$\boxed{I_{xx} = \frac{bh^3}{12}}$$

Similarly, we may find, $I_{yy} = \frac{hb^3}{12}$

∴ Polar moment of inertia (J) = $I_{xx} + I_{yy} = \frac{bh^3}{12} + \frac{hb^3}{12}$

If we want to know the ***MOI about an axis NN*** passing through the bottom edge or top edge.

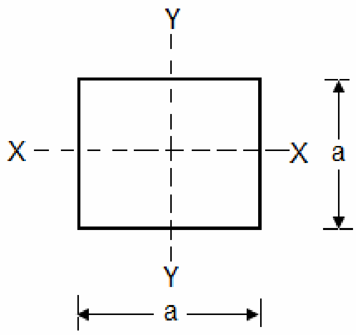


Axis XX and NN are parallel and at a distance h/2.

Therefore $I_{NN} = I_{xx}$ + **Area** $\times$ **(distance)** ^{2}

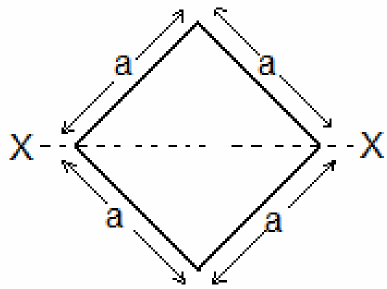
$$= \frac{bh^3}{12} + b \times h \times \left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$

Case-I: Square area



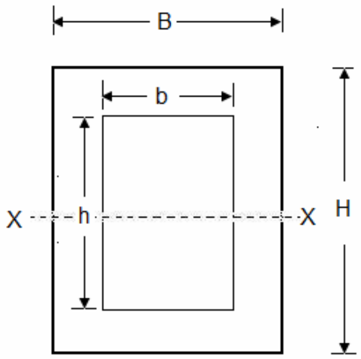
$$I_{xx} = \frac{a^4}{12}$$

Case-II: Square area with diagonal as axis



$$I_{xx} = \frac{a^4}{12}$$

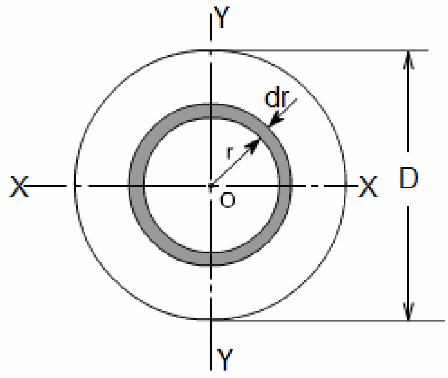
Case-III: Rectangular area with a centrally rectangular hole



Moment of inertia of the area = moment of inertia of BIG rectangle – moment of inertia of SMALL rectangle

$$I_{xx} = \frac{BH^3}{12} - \frac{bh^3}{12}$$

(ii) MOI of a Circular area



The moment of inertia about axis XX this passes through the centroid. It is very easy to find polar moment of inertia about point 'O'. Take an element of width 'dr' at a distance 'r' from centre. Therefore, the moment of inertia of this element about polar axis

$$d(J) = d(I_{xx} + I_{yy}) = \text{area of ring} \times (\text{radius})^2$$

$$\text{or } d(J) = 2\pi r dr \times r^2$$

Integrating both side we get

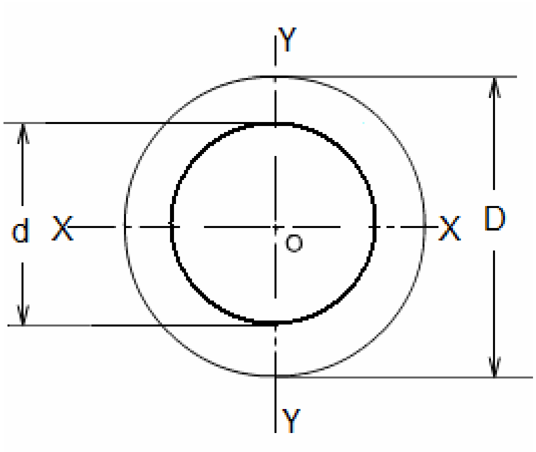
$$J = \int_0^R 2\pi r^3 dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32}$$

Due to summetry $I_{xx} = I_{yy}$

Therefore, $I_{xx} = I_{yy} = \dfrac{J}{2} = \dfrac{\pi D^4}{64}$

$$\boldsymbol{I_{xx} = I_{yy} = \frac{\pi D^4}{64} \quad \text{and } J = \frac{\pi D^4}{32}}$$

Case-I: Moment of inertia of a circular area with a concentric hole.



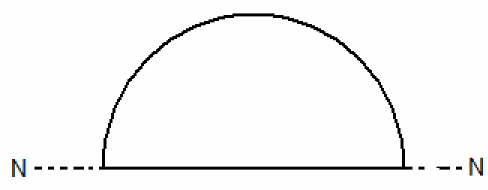
Moment of inertia of the area = moment of inertia of BIG circle – moment of inertia of SMALL circle.

$$I_{xx} = I_{yy} = \frac{\pi D^4}{64} - \frac{\pi d^4}{64}$$

$$= \frac{\pi}{64}(D^4 - d^4)$$

$$\text{and } J = \frac{\pi}{32}(D^4 - d^4)$$

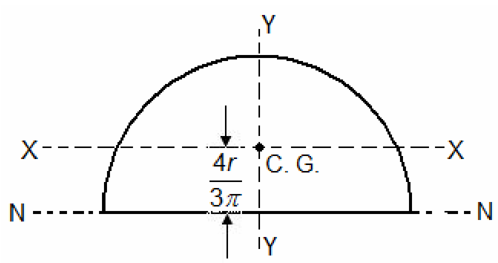
Case-II: Moment of inertia of a semi-circular area.



$$I_{NN} = \frac{1}{2} \text{ of the momemt of total circular lamina}$$

$$= \frac{1}{2} \times \left(\frac{\pi D^4}{64}\right) = \frac{\pi D^4}{128}$$

We know that distance of CG from base is $\dfrac{4r}{3\pi} = \dfrac{2D}{3\pi} = h(\text{say})$



i.e. distance of parallel axis XX and NN is (h)

$\therefore$ According to parallel axis theory

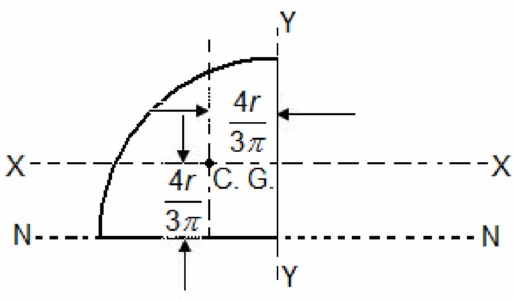
$$I_{NN} = I_G + \text{Area} \times (\text{distance})^2$$

$$\text{or } \frac{\pi D^4}{128} = I_{xx} + \frac{1}{2}\left(\frac{\pi D^2}{4}\right) \times (h)^2$$

$$\text{or } \frac{\pi D^4}{128} = I_{xx} + \frac{1}{2} \times \left(\frac{\pi D^2}{4}\right) \times \left(\frac{2D}{3\pi}\right)$$

or $$\boxed{I_{xx} = 0.11R^4}$$

Case – III: Quarter circle area



I_{XX} = one half of the moment of Inertia of the Semi-circular area about XX.

$$I_{XX} = \frac{1}{2} \times (0.11R^4) = 0.055\ R^4$$

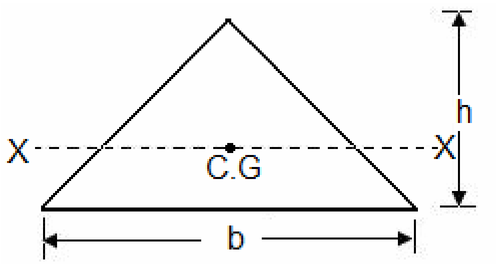
$$\boxed{I_{XX} = 0.055\,R^4}$$

I_{NN} = one half of the moment of Inertia of the Semi-circular area about NN.

$$\therefore\ I_{NN} = \frac{1}{2} \times \frac{\pi D^4}{64} = \frac{\pi D^4}{128}$$

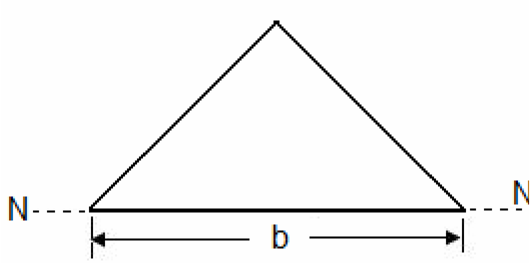
(iii) Moment of Inertia of a Triangular area

(a) Moment of Inertia of a Triangular area of a axis XX parallel to base and passes through C.G.



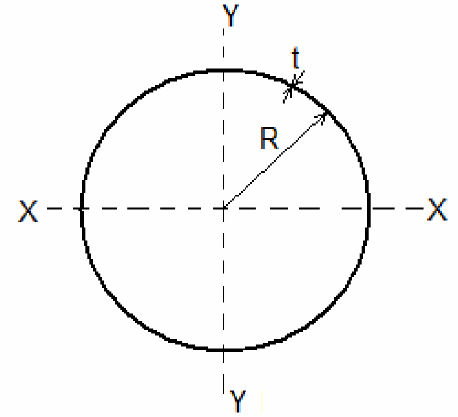
$$\boxed{I_{XX} = \frac{bh^3}{36}}$$

(b) Moment of inertia of a triangle about an axis passes through base



$$\boxed{I_{NN} = \frac{bh^3}{12}}$$

(iv) Moment of inertia of a thin circular ring:



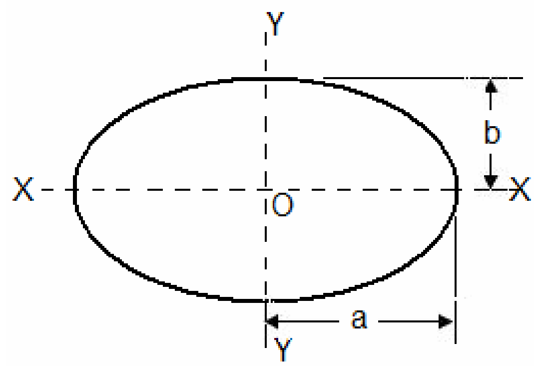
Polar moment of Inertia

$$(J) = R^2 \times \text{area of whole ring}$$

$$= R^2 \times 2\pi R t = 2\pi R^3 t$$

$$I_{XX} = I_{YY} = \frac{J}{2} = \pi R^3 t$$

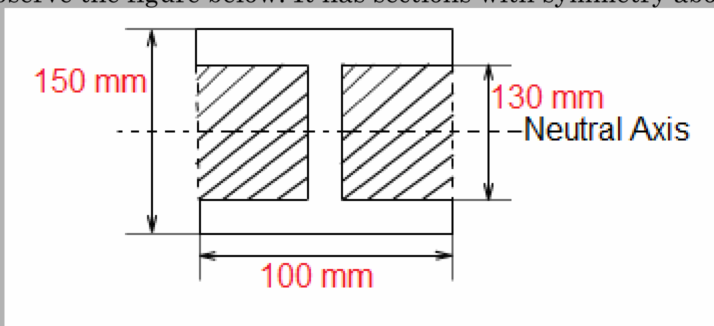
(v) Moment of inertia of a elliptical area



$$I_{XX} = \frac{\pi a b^3}{4}$$

Let us take an example: An I-section beam of 100 mm wide, 150 mm depth flange and web of thickness 20 mm is used in a structure of length 5 m. Determine the Moment of Inertia (of area) of cross-section of the beam.

Answer: Carefully observe the figure below. It has sections with symmetry about the neutral axis.



We may use standard value for a rectangle about an axis passes through centroid. i.e. $I = \frac{bh^3}{12}$.

The section can thus be divided into convenient rectangles for each of which the neutral axis passes the centroid.

$$I_{Beam} = I_{Rectangle} - I_{Shaded\ area}$$

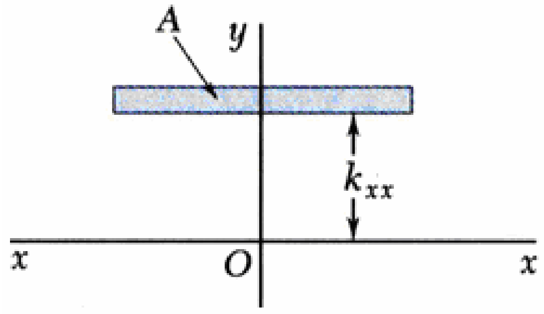
$$= \left[\frac{0.100 \times (0.150)^3}{12} - 2 \times \frac{0.40 \times 0.130^3}{12}\right] m^4$$

$$= 1.183 \times 10^{-4}\ m^4$$

3.9 Radius of gyration

Consider area A with moment of inertia I_{xx}. Imagine that the area is concentrated in a thin strip parallel to

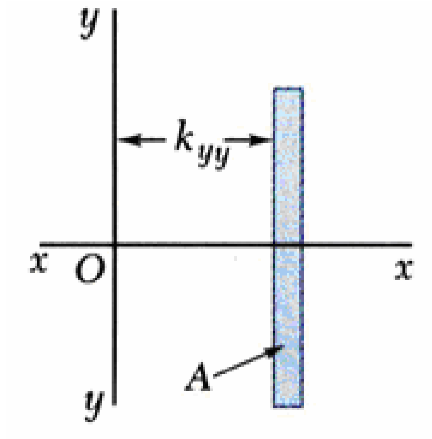
the x axis with equivalent I_{xx}.



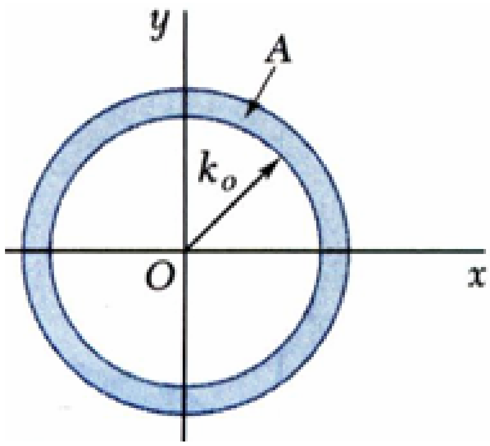
$$I_{xx} = k_{xx}^2 A \text{ or } \boxed{k_{xx} = \sqrt{\frac{I_{xx}}{A}}}$$

k_{xx} =*radius of gyration* with respect to the x axis.

Similarly



$$I_{yy} = k_{yy}^2 A \text{ or } \boxed{k_{yy} = \sqrt{\frac{I_{yy}}{A}}}$$

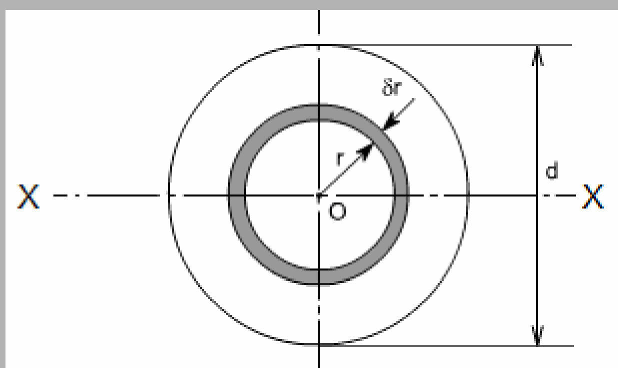


$$J = k_o^2 A \text{ or } \boxed{k_o = \sqrt{\frac{J}{A}}}$$

$$k_o^2 = k_{xx}^2 + k_{yy}^2$$

Let us take an example: Find radius of gyration for a circular area of diameter 'd' about central axis.

Answer:



We know that, $I_{xx} = K_{xx}^2 A$

$$\text{or } K_{XX} = \sqrt{\frac{I_{XX}}{A}} = \sqrt{\frac{\frac{\pi d^4}{64}}{\frac{\pi d^2}{4}}} = \frac{d}{4}$$

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Moment of Inertia (Second moment of an area)

GATE-1. The second moment of a circular area about the diameter is given by (D is the diameter) [GATE-2003]

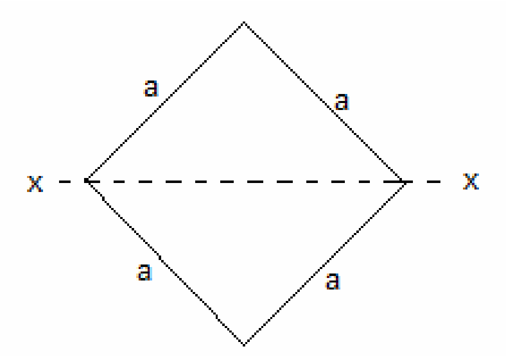
(a) $\frac{\pi D^4}{4}$ (b) $\frac{\pi D^4}{16}$ (c) $\frac{\pi D^4}{32}$ (d) $\frac{\pi D^4}{64}$

GATE-1. Ans. (d)

GATE-2. The area moment of inertia of a square of size 1 unit about its diagonal is: [GATE-2001]

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{12}$ (d) $\frac{1}{6}$

GATE-2. Ans. (c) $I_{xx} = \frac{a^4}{12} = \frac{(1)^4}{12}$

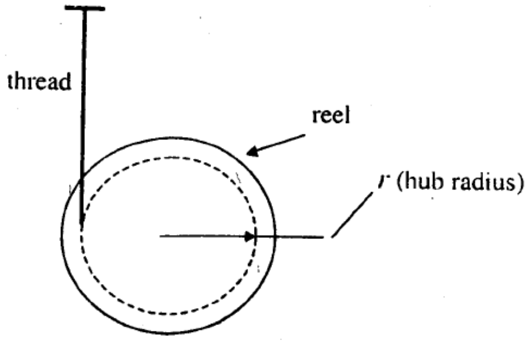


Radius of Gyration

Data for Q3–Q4 are given below. Solve the problems and choose correct answers.

A reel of mass "m" and radius of gyration "k" is rolling down smoothly from rest with one end of the thread wound on it held in the ceiling as depicted in the figure. Consider the thickness of the thread and its mass negligible in comparison with the radius "r" of the hub and the reel mass "m". Symbol "g" represents the acceleration due to gravity.

[GATE-2003]

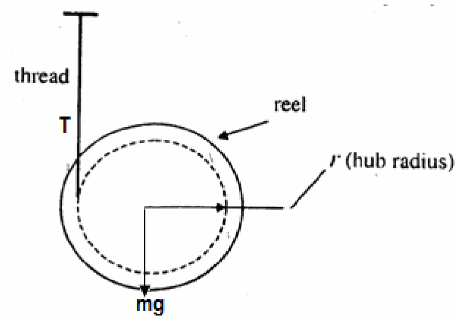


GATE-3. The linear acceleration of the reel is:

(a) $\frac{gr^2}{(r^2+k^2)}$ (b) $\frac{gk^2}{(r^2+k^2)}$ (c) $\frac{grk}{(r^2+k^2)}$ (d) $\frac{mgr^2}{(r^2+k^2)}$

GATE-3. Ans. (a) For downward linear motion mg – T = mf, where f = linear tangential acceleration = rα, α = rotational acceleration. Considering rotational motion $Tr = I\alpha$.

or, T = $mk^2 \times \frac{f}{r^2}$ therefore mg – T = mf gives f = $\frac{gr^2}{(r^2+k^2)}$



GATE-4. The tension in the thread is:

(a) $\frac{mgr^2}{(r^2+k^2)}$ (b) $\frac{mgrk}{(r^2+k^2)}$ (c) $\frac{mgk^2}{(r^2+k^2)}$ (d) $\frac{mg}{(r^2+k^2)}$

GATE-4. Ans. (c) $T = mk^2 \times \frac{f}{r^2} = mk^2 \times \frac{gr^2}{r^2(r^2+k^2)} = \frac{mgk^2}{(r^2+k^2)}$

Previous 20-Years IES Questions

Centroid

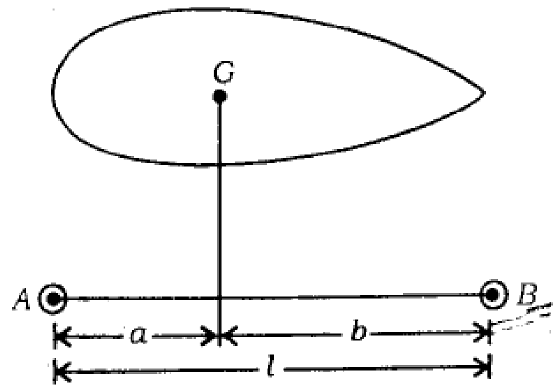
IES-1. Assertion (A): Inertia force always acts through the centroid of the body and is directed opposite to the acceleration of the centroid. [IES-2001]
Reason (R): It has always a tendency to retard the motion.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-1. Ans. (c) It has always a tendency to oppose the motion not retard. If we want to retard a motion then it will wand to accelerate.

Radius of Gyration

IES-2. Figure shows a rigid body of mass m having radius of gyration k about its centre of gravity. It is to be replaced by an equivalent dynamical system of two masses placed at A and B. The mass at A should be:



(a) $\frac{a \times m}{a+b}$ (b) $\frac{b \times m}{a+b}$

(c) $\frac{m}{3} \times \frac{a}{b}$ (d) $\frac{m}{2} \times \frac{b}{a}$

[IES-2003]

IES-2. Ans. (b)

IES-3. Force required to accelerate a cylindrical body which rolls without slipping on a horizontal plane (mass of cylindrical body is m, radius of the cylindrical surface in contact with plane is r, radius of gyration of body is k and acceleration of the body is a) is: [IES-2001]

(a) $m\left(k^2/r^2+1\right).a$ (b) $\left(mk^2/r^2\right).a$ (c) $mk^2.a$ (d) $\left(mk^2/r+1\right).a$

IES-3. Ans. (a)

IES-4. A body of mass m and radius of gyration k is to be replaced by two masses m_1 and m_2 located at distances h_1 and h_2 from the CG of the original body. An equivalent dynamic system will result, if [IES-2001]

(a) $h_1 + h_2 = k$ (b) $h_1^2 + h_2^2 = k^2$ (c) $h_1 h_2 = k^2$ (d) $\sqrt{h_1 h_2} = k^2$

IES-4. Ans. (c)

Previous 20-Years IAS Questions

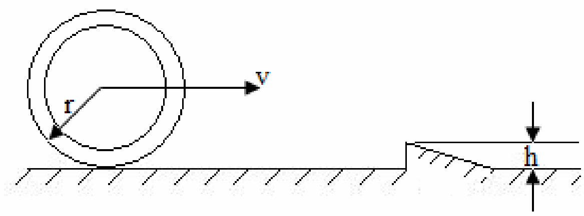
Radius of Gyration

IAS-1. A wheel of centroidal radius of gyration 'k' is rolling on a horizontal surface with constant velocity. It comes across an obstruction of height 'h' Because of its rolling speed, it just overcomes the obstruction. To determine v, one should use the principle (s) of conservation of [IAS 1994]

(a) Energy (b) Linear momentum

(c) Energy and linear momentum (d) Energy and angular momentum

IAS-1. Ans. (a)



Previous Conventional Questions with Answers

Conventional Question IES-2004

Question: **When are I-sections preferred in engineering applications? Elaborate your answer.**

Answer: I-section has large section modulus. It will reduce the stresses induced in the material. Since I-section has the considerable area are far away from the natural so its section modulus increased.

4. Bending Moment and Shear Force Diagram

Theory at a Glance (for IES, GATE, PSU)

4.1 Shear Force and Bending Moment

At first we try to understand what shear force is and what is bending moment?

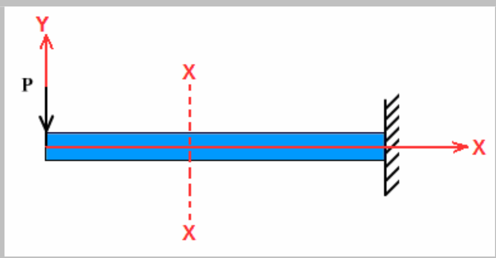
We will not introduce any other co-ordinate system. We use general co-ordinate axis as shown in the figure. This system will be followed in shear force and bending moment diagram and in deflection of beam. Here downward direction will be negative i.e. negative Y-axis. Therefore downward deflection of the beam will be treated as negative.

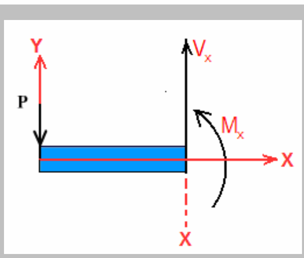
We use above Co-ordinate system

Some books fix a co-ordinate axis as shown in the following figure. Here downward direction will be positive i.e. positive Y-axis. Therefore downward deflection of the beam will be treated as positive. As beam is generally deflected in downward directions and this co-ordinate system treats downward deflection is positive deflection.

Some books use above co-ordinate system

Consider a cantilever beam as shown subjected to external load 'P'. If we imagine this beam to be cut by a section X-X, we see that the applied force tend to displace the left-hand portion of the beam relative to the right hand portion, which is fixed in the wall. This tendency is resisted by internal forces between the two parts of the beam. At the cut section a resistance shear force (V_x) and a bending moment (M_x) is induced. This resistance shear force and the bending moment at the cut section is shown in the left hand and right hand portion of the cut beam. Using the three equations of equilibrium



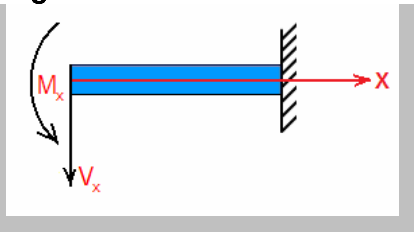


$\sum F_x = 0$, $\sum F_y = 0$ and $\sum M_i = 0$

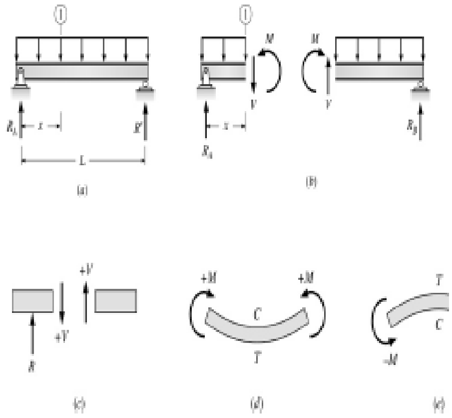
We find that $V_x = -P$ and $M_x = -P.x$

In this chapter we want to show pictorially the

variation of shear force and bending moment in a beam as a function of 'x' measured from one end of the beam.



Shear Force (V) ≡ equal in magnitude but opposite in direction to the algebraic sum (resultant) of the components in the direction perpendicular to the axis of the beam of all external loads and support reactions acting on either side of the section being considered.



Bending Moment (M) equal in magnitude but opposite in direction to the algebraic sum of the moments about (the centroid of the cross section of the beam) the section of all external loads and support reactions acting on either side of the section being considered.

What are the benefits of drawing shear force and bending moment diagram?

The benefits of drawing a variation of shear force and bending moment in a beam as a function of 'x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment. The shear force and bending moment diagram gives a clear picture in our mind about the variation of SF and BM throughout the entire section of the beam.

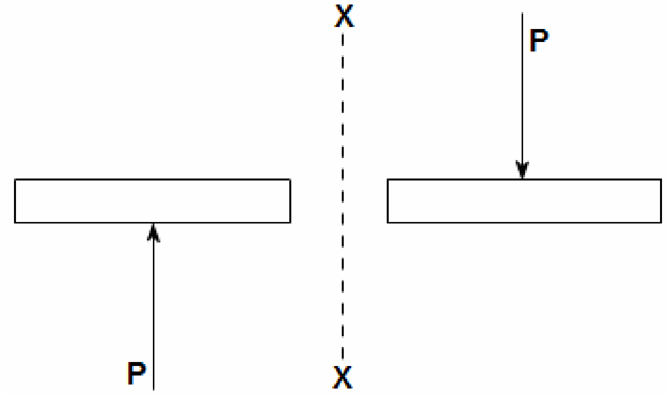
Further, the determination of value of bending moment as a function of 'x' becomes very important so as to determine the value of deflection of beam subjected to a given loading where we will use the formula, $EI\frac{d^2y}{dx^2} = M_x$.

4.2 Notation and sign convention

- **Shear force (V)**

Positive Shear Force

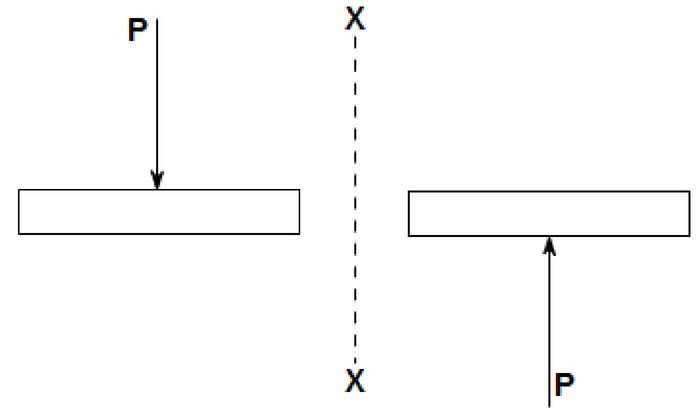
A shearing force having a downward direction to the right hand side of a section or upwards to the left hand of the section will be taken as 'positive'. It is the usual sign conventions to be followed for the shear force. In some book followed totally opposite sign convention.



The upward direction shearing force which is on the ***left hand*** of the section XX is ***positive*** shear force.	The downward direction shearing force which is on the ***right hand*** of the section XX is ***positive*** shear force.

Negative Shear Force

A shearing force having an upward direction to the right hand side of a section or downwards to the left hand of the section will be taken as 'negative'.

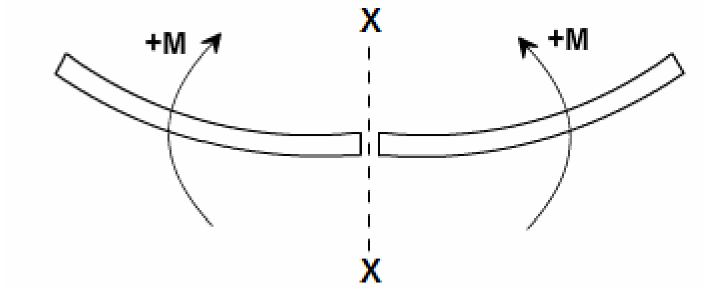


The downward direction shearing force which is on the ***left hand*** of the section XX is ***negative*** shear force.	The upward direction shearing force which is on the ***right hand*** of the section XX is ***negative*** shear force.

- ## Bending Moment (M)

Positive Bending Moment

A bending moment causing concavity upwards will be taken as 'positive' and called as sagging bending moment.



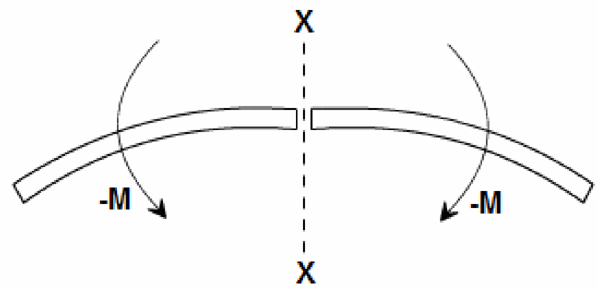
Sagging

If the bending moment of the ***left hand*** of the section XX is ***clockwise*** then it is a ***positive*** bending moment.

If the bending moment of the ***right hand*** of the section XX is ***anti-clockwise*** then it is a ***positive*** bending moment.

A bending moment causing concavity upwards will be taken as 'positive' and called as sagging bending moment.

Negative Bending Moment



Hogging

If the bending moment of the ***left hand*** of the section XX is ***anti-clockwise*** then it is a ***positive*** bending moment.

If the bending moment of the ***right hand*** of the section XX is ***clockwise*** then it is a ***positive*** bending moment.

A bending moment causing convexity upwards will be taken as 'negative' and called as hogging bending moment.

Way to remember sign convention

- **Remember in the *Cantilever beam both Shear force and BM are negative (–ive).***

4.3 Relation between S.F (V_x), B.M. (M_x) & Load (w)

- $\dfrac{dV_x}{dx} = -w\ (\text{load})$ The value of the distributed load at any point in the beam is equal to the slope of the shear force curve. (Note that the sign of this rule may change depending on the sign convention used for the external distributed load).

- $\dfrac{dM_x}{dx} = V_x$ The value of the shear force at any point in the beam is equal to the slope of the bending moment curve.

4.4 Procedure for drawing shear force and bending moment diagram

Construction of shear force diagram

- From the loading diagram of the beam constructed shear force diagram.
- First determine the reactions.
- Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.
- The shear force curve is continuous unless there is a point force on the beam. The curve then "jumps" by the magnitude of the point force (+ for upward force).
- When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam). No shear force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations. i.e. The shear force will be zero at each end of the beam unless a point force is applied at the end.

Construction of bending moment diagram

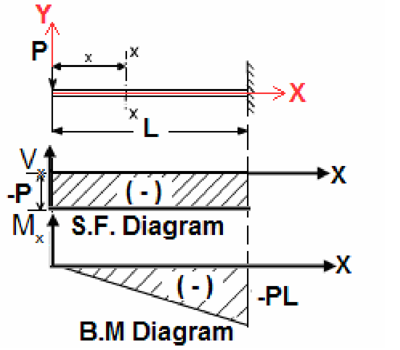
- The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams using proper sign convention.
- The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.
- The bending moment curve is continuous unless there is a point moment on the beam. The curve then "jumps" by the magnitude of the point moment (+ for CW moment).
- We know that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. We also know that $dM/dx = V_x$ therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero.
- The bending moment will be zero at each free or pinned end of the beam. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction.

4.5 Different types of Loading and their S.F & B.M Diagram

(i) A Cantilever beam with a concentrated load 'P' at its free end.

Shear force:

At a section a distance x from free end consider the forces to the left, then ***(V_x) = - P (for all values of x)*** negative in sign i.e. the shear force to the left of the x-section are in downward direction and therefore negative.



S.F and B.M diagram

Bending Moment:

Taking moments about the section gives (obviously to the left of the section) $M_x = -P.x$ (negative sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as negative according to the sign convention) so that the ***maximum*** bending moment occurs at the fixed end i.e. $M_{max} = -PL$ (at x = L)

(ii) A Cantilever beam with uniformly distributed load over the whole length

When a cantilever beam is subjected to a uniformly distributed load whose intensity is given w /unit length.

Shear force:

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$V_x = -w.x$ for all values of 'x'.

At x = 0, $V_x = 0$

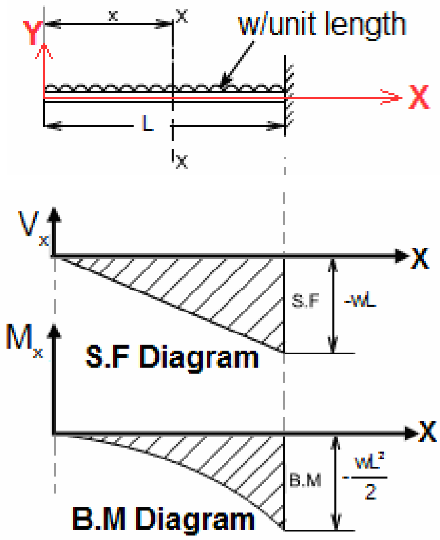
At x = L, $V_x = -wL$ (i.e. Maximum at fixed end)

Plotting the equation $V_x = -w.x$, we get a straight line because it is a equation of a straight line y (V_x) = m(- w) .x

Bending Moment:

Bending Moment at XX is obtained by treating the load to the left of XX as a concentrated load of the same value (w.x) acting through the centre of gravity at x/2.

Therefore, the bending moment at any cross-section XX is



S.F and B.M diagram

$$M_x = (-w.x).\frac{x}{2} = -\frac{w.x^2}{2}$$

Therefore *the variation of bending moment is according to* ***parabolic law.***

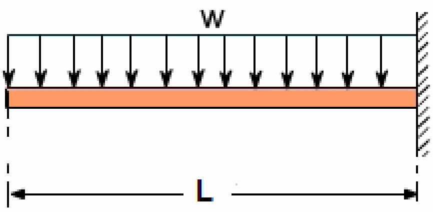
The extreme values of B.M would be

at x = 0, $M_x = 0$

and x = L, $M_x = -\frac{wL^2}{2}$

Maximum bending moment, $M_{max} = \dfrac{wL^2}{2}$ at fixed end

Another way to describe a cantilever beam with uniformly distributed load (UDL) over it's whole length.



(iii) A Cantilever beam loaded as shown below draw its S.F and B.M diagram

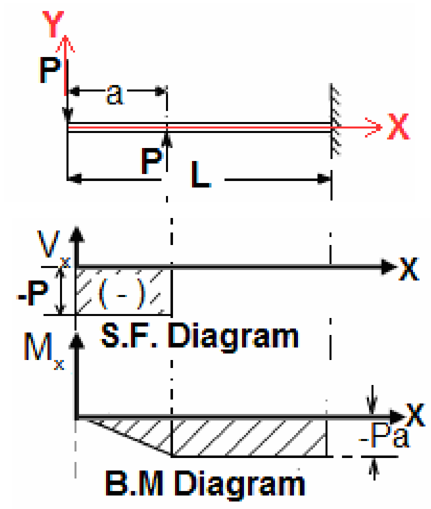
In the region 0 < x < a

Following the same rule as followed previously, we get

$V_x = -P;\ \text{and } M_x = -P.x$

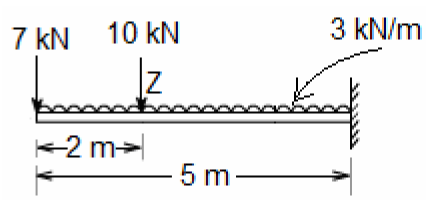
In the region a < x < L

$V_x = -P + P = 0;\ \text{and } M_x = -P.x + P(x-a) = P.a$



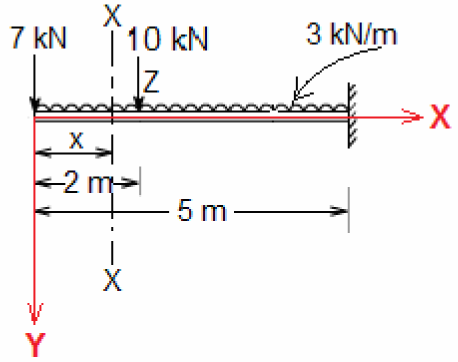
S.F and B.M diagram

(iv) ***Let us take an example:*** Consider a cantilever bean of 5 m length. It carries a uniformly distributed load 3 KN/m and a concentrated load of 7 kN at the free end and 10 kN at 3 meters from the fixed end.



Draw SF and BM diagram.

Answer: In the region 0 < x < 2 m



Consider any cross section XX at a distance x from free end.

Shear force (V_x) = -7- 3x

So, the variation of shear force is linear.

at x = 0, V_x = -7 kN

at x = 2 m , $V_x = -7 - 3\times 2 = -13$ kN

at point Z $V_x = -7 - 3\times 2 - 10 = -23$ kN

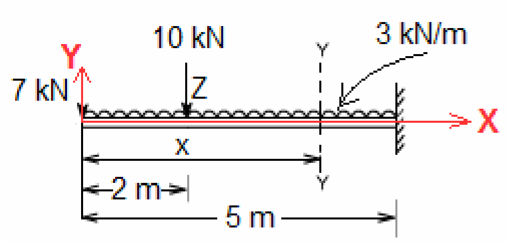
Bending moment (M_x) = $-7x - (3x).\dfrac{x}{2} = -\dfrac{3x^2}{2} - 7x$

So, the variation of bending force is parabolic.

at x = 0, $M_x = 0$

at x = 2 m, $M_x = -7\times 2 - (3\times 2)\times\dfrac{2}{2} = -20$ kNm

In the region 2 m < x < 5 m



Consider any cross section YY at a distance x from free end

Shear force (V_x) = $-7 - 3x - 10 = -17 - 3x$

So, the variation of shear force is linear.

at x = 2 m, $V_x = -23$ kN

at x = 5 m, $V_x = -32$ kN

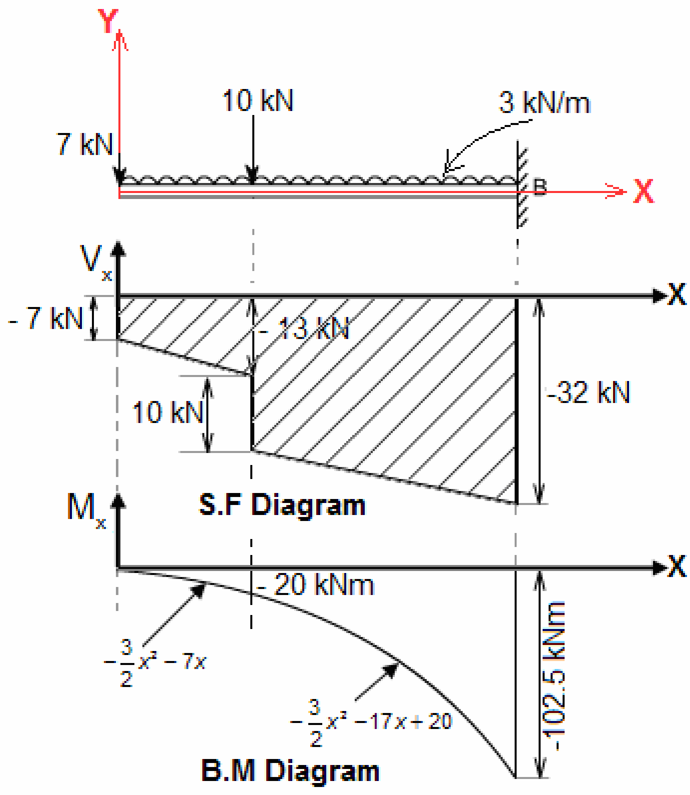
Bending moment (M_x) = $-7x - (3x)\times\left(\dfrac{x}{2}\right) - 10(x-2)$

$$= -\frac{3}{2}x^2 - 17x + 20$$

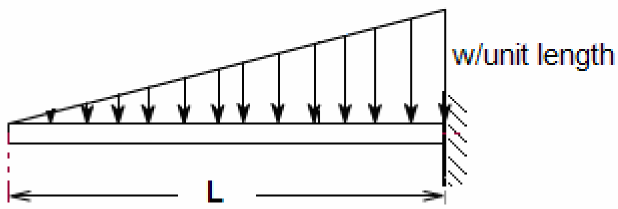
So, the variation of bending force is parabolic.

at x = 2 m, $M_x = -\dfrac{3}{2}\times 2^2 - 17\times 2 + 20 = -20$ kNm

at x = 5 m, $M_x = -102.5$ kNm



(v) A Cantilever beam carrying uniformly varying load from zero at free end and w/unit length at the fixed end



Consider any cross-section XX which is at a distance of x from the free end.

At this point load (w_x) = $\frac{w}{L}.x$

Therefore total load (W) $= \int_0^L w_x dx = \int_0^L \frac{w}{L}.xdx = \frac{wL}{2}$

Shear force $(\mathbf{V_x})$ = area of ABC (load triangle)

$$= -\frac{1}{2}.\left(\frac{w}{L}x\right).x = -\frac{wx^2}{2L}$$

$\therefore$ The shear force variation is parabolic.

at x = 0, $V_x = 0$

at x = L, $V_x = -\frac{WL}{2}$ i.e. Maximum Shear force $(V_{max}) = \frac{-WL}{2}$ at fixed end

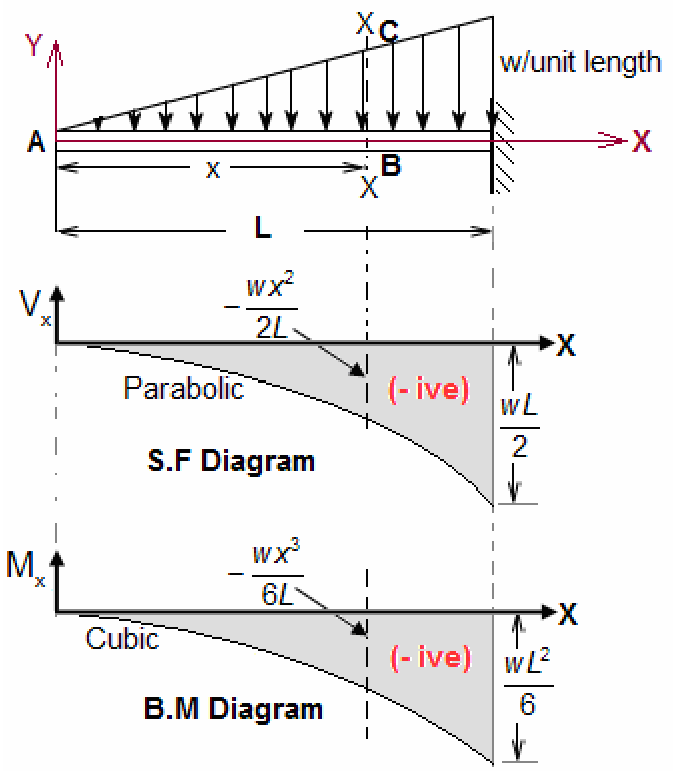
Bending moment (M_x) = load × distance from centroid of triangle ABC

$$= -\frac{wx^2}{2L}.\left(\frac{2x}{3}\right) = -\frac{wx^3}{6L}$$

∴ The bending moment variation is cubic.

at x= 0, $M_x = 0$

at x = L, $M_x = -\frac{wL^2}{6}$ i.e. Maximum Bending moment $(M_{max}) = \frac{wL^2}{6}$ at fixed end.



Alternative way : (Integration method)

We know that $\frac{d(V_x)}{dx} = -\text{load} = -\frac{w}{L}.x$

$$\text{or } d(V_x) = -\frac{w}{L}.x\ .dx$$

Integrating both side

$$\int_0^{V_x} d(V_x) = -\int_0^x \frac{w}{L}.\ x\ .dx$$

$$\text{or } V_x = -\frac{w}{L}.\frac{x^2}{2}$$

Again we know that

$$\frac{d(M_x)}{dx} = V_x = -\frac{wx^2}{2L}$$

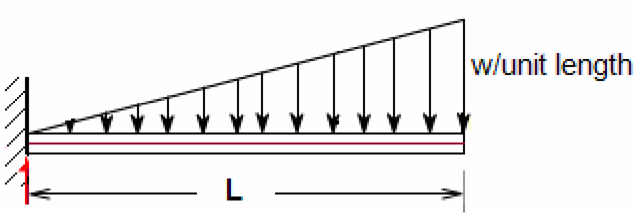
$$\text{or} \quad d(M_x) = -\frac{wx^2}{2L}dx$$

Integrating both side we get $\left(\text{at } x=0, M_x=0\right)$

$$\int_0^{M_x} d(M_x) = -\int_0^x \frac{wx^2}{2L}.dx$$

$$\text{or } M_x = -\frac{w}{2L}\times\frac{x^3}{3} = -\frac{wx^3}{6L}$$

(vi) A Cantilever beam carrying gradually varying load from zero at fixed end and w/unit length at the free end



Considering equilibrium we get, $M_A = \frac{wL^2}{3}$ and Reaction $\left(R_A\right) = \frac{wL}{2}$

Considering any cross-section XX which is at a distance of x from the fixed end.

At this point load $(W_x) = \frac{W}{L}.x$

Shear force $\left(\mathbf{V_x}\right) = R_A - \text{area of triangle ANM}$

$$= \frac{wL}{2} - \frac{1}{2}.\left(\frac{w}{L}.x\right).x = +\frac{wL}{2} - \frac{wx^2}{2L}$$

$\therefore$ The shear force variation is parabolic.

at $x = 0,\ V_x = +\frac{wL}{2}$ i.e. Maximum shear force, $V_{max} = +\frac{wL}{2}$

at $x = L,\ V_x = 0$

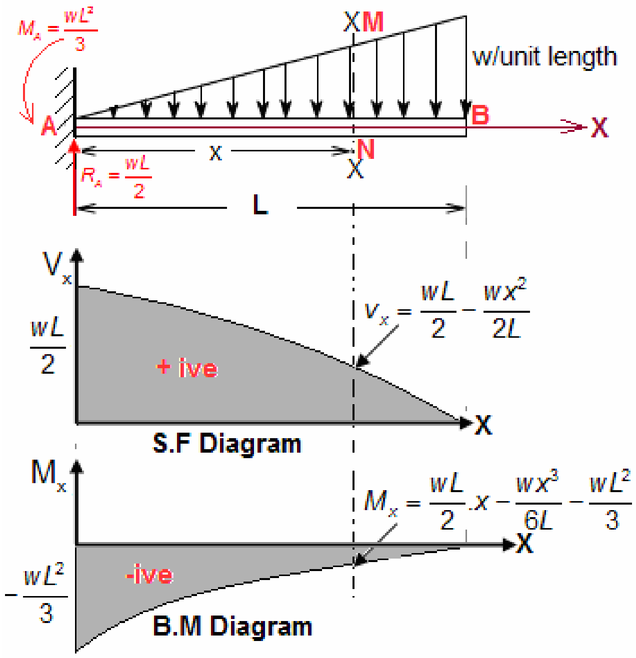
Bending moment $\left(\mathbf{M_x}\right) = R_A.x - \frac{wx^2}{2L}.\frac{2x}{3} - M_A$

$$= \frac{wL}{2}.x - \frac{wx^3}{6L} - \frac{wL^2}{3}$$

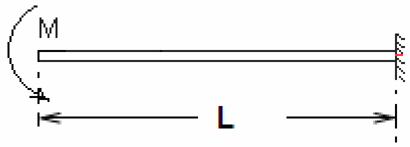
$\therefore$ The bending moment variation is cubic

at $x = 0,\ M_x = -\frac{wL^2}{3}$ i.e.Maximum B.M. $\left(M_{max}\right) = -\frac{wL^2}{3}$.

at $x = L,\ M_x = 0$



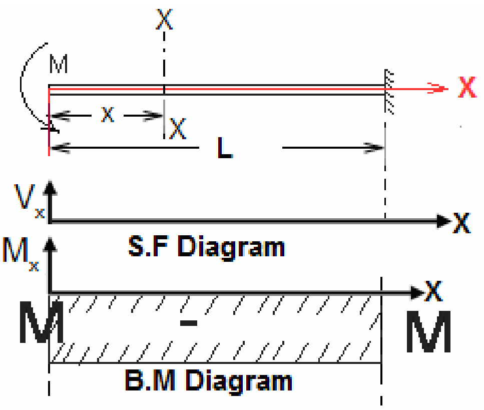
(vii) A Cantilever beam carrying a moment M at free end



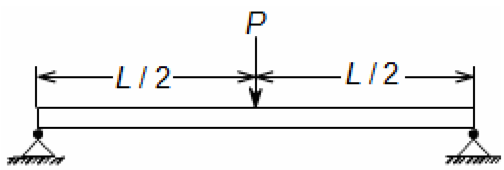
Consider any cross-section XX which is at a distance of x from the free end.

Shear force: $V_x = 0$ at any point.

Bending moment (M_x) = -M at any point, i.e. Bending moment is constant throughout the length.



(viii) A Simply supported beam with a concentrated load 'P' at its mid span.



Considering equilibrium we get, $R_A = R_B = \frac{P}{2}$

Now consider any cross-section XX which is at a distance of x from left end A and section YY at a distance from left end A, as shown in figure below.

Shear force: In the region 0 < x < L/2

$V_x = R_A = + P/2$ (it is constant)

In the region L/2 < x < L

$V_x = R_A - P = \frac{P}{2} - P = - P/2$ (it is constant)

Bending moment: In the region 0 < x < L/2

$M_x = \frac{P}{2}.x$ (its variation is linear)

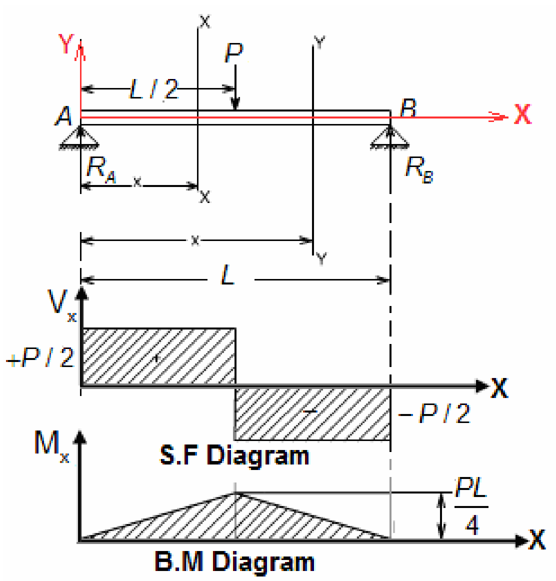
at x = 0, $M_x = 0$ and at x = L/2 $M_x = \frac{PL}{4}$ i.e. maximum

Maximum bending moment, $\boxed{M_{max} = \frac{PL}{4}}$ at x = L/2 (at mid-point)

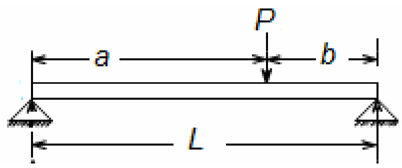
In the region L/2 < x < L

$M_x = \frac{P}{2}.x - P(x - L/2) = \frac{PL}{2} - \frac{P}{2}.x$ (its variation is linear)

at x = L/2 , $M_x = \frac{PL}{4}$ and at x = L, $M_x = 0$



(ix) A Simply supported beam with a concentrated load 'P' is *not* at its mid span.



Considering equilibrium we get, $R_A = \frac{Pb}{L}$ and $R_B = \frac{Pa}{L}$

Now consider any cross-section XX which is at a distance x from left end A and another section YY at a distance x from end A as shown in figure below.

Shear force: In the range 0 < x < a

$$V_x = R_A = +\frac{Pb}{L} \qquad \text{(it is constant)}$$

In the range a < x < L

$$V_x = R_A - P = -\frac{Pa}{L} \qquad \text{(it is constant)}$$

Bending moment: In the range 0 < x < a

$$M_x = +R_A.x = \frac{Pb}{L}.x \qquad \text{(it is variation is linear)}$$

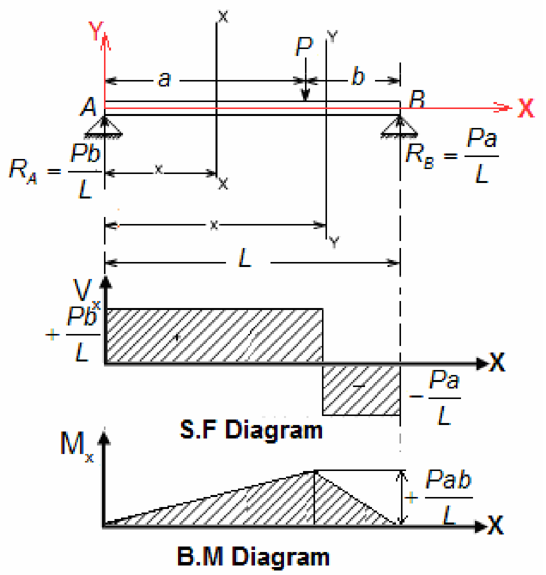
$$\text{at } x = 0, M_x = 0 \quad \text{and} \quad \text{at } x = a, M_x = \frac{Pab}{L} \qquad \text{(i.e. maximum)}$$

In the range a < x < L

$$M_x = R_A.x - P(x-a) = \frac{Pb}{L}.x - P.x + Pa \quad (\text{Put } b = L - a)$$

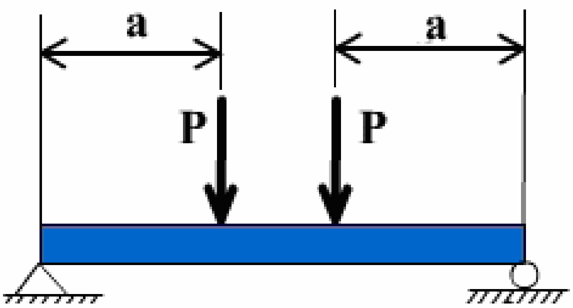
$$= Pa\ (1 - Pa\left(1 - \frac{x}{L}\right))$$

$$\text{at } x = a, \ M_x = \frac{Pab}{L} \quad \text{and} \quad \text{at } x = L, \ M_x = 0$$



(x) A Simply supported beam with two concentrated load 'P' from a distance 'a' both end.

The loading is shown below diagram

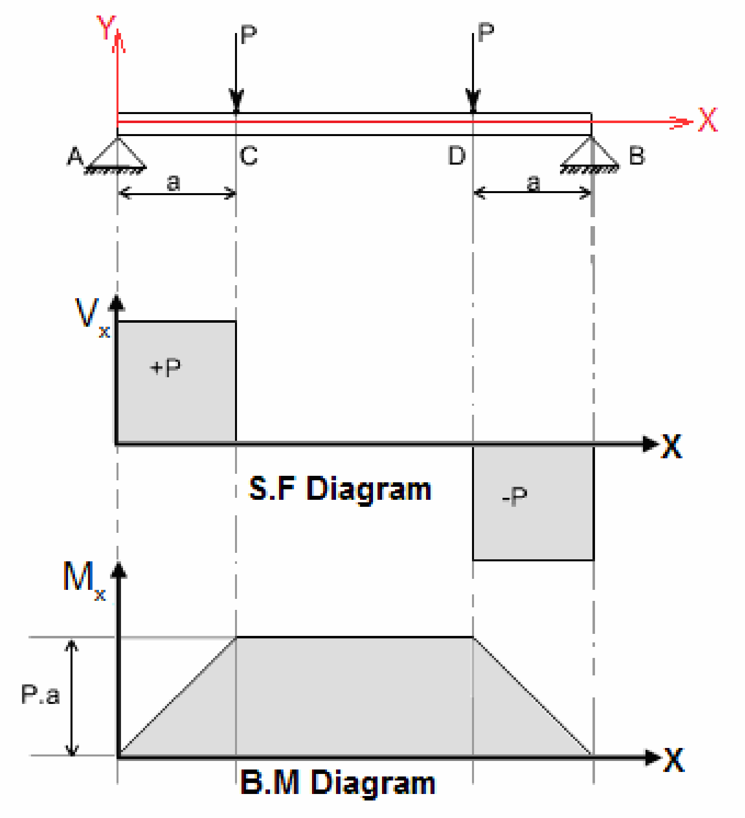


Take a section at a distance x from the left support. This section is applicable for any value of x just to the left of the applied force P. The shear, remains constant and is +P. The bending moment varies linearly from the support, reaching a maximum of +Pa.

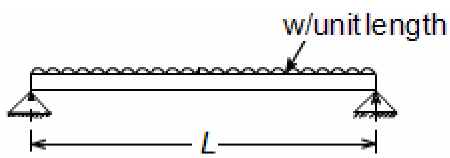
A section applicable anywhere between the two applied forces. Shear force is not necessary to maintain equilibrium of a segment in this part of the beam. Only a constant bending moment of +Pa must be resisted by the beam in this zone.

Such a state of bending or flexure is called ***pure bending.***

Shear and bending-moment diagrams for this loading condition are shown below.



(xi) A Simply supported beam with a uniformly distributed load (UDL) through out its length



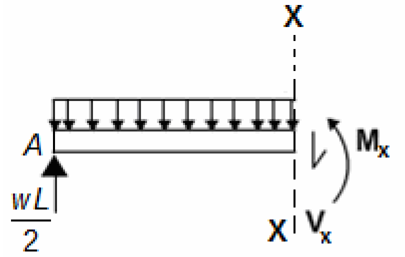
We will solve this problem by following two alternative ways.

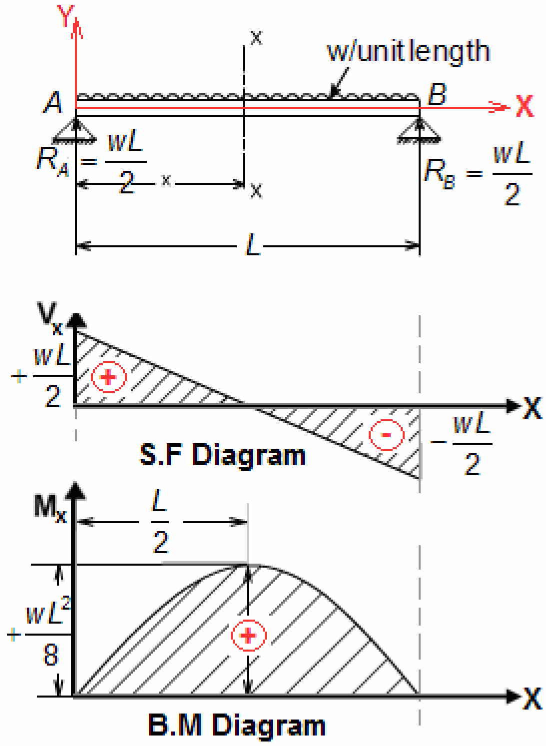
(a) By Method of Section

Considering equilibrium we get $R_A = R_B = \dfrac{wL}{2}$

Now Consider any cross-section XX which is at a distance x from left end A.

Then the section view





Shear force: $V_x = \frac{wL}{2} - wx$

(i.e. S.F. variation is linear)

at $x = 0$, $V_x = \frac{wL}{2}$

at $x = L/2$, $V_x = 0$

at $x = L$, $V_x = -\frac{wL}{2}$

Bending moment: $M_x = \frac{wL}{2}.x - \frac{wx^2}{2}$

(i.e. B.M. variation is parabolic)

at $x = 0$, $M_x = 0$

at $x = L$, $M_x = 0$

Now we have to determine maximum bending moment and its position.

For maximum B.M: $\frac{d(M_x)}{dx} = 0 \quad i.e.\ V_x = 0 \quad \left[\because \frac{d(M_x)}{dx} = V_x\right]$

or $\frac{wL}{2} - wx = 0 \quad or \quad x = \frac{L}{2}$

Therefore, maximum bending moment, $\boxed{M_{max} = \frac{wL^2}{8}}$ **at x = L/2**

(a) By Method of Integration

Shear force:

We know that, $\frac{d(V_x)}{dx} = -w$

$or \qquad d(V_x) = -wdx$

Integrating both side we get (at x =0, $V_x = \frac{wL}{2}$)

$$\int_{+\frac{wL}{2}}^{V_x} d(V_x) = -\int_0^x w dx$$

$$or\ V_x - \frac{wL}{2} = -wx$$

$$or\ V_x = \frac{wL}{2} - wx$$

Bending moment:

We know that, $\frac{d(M_x)}{dx} = V_x$

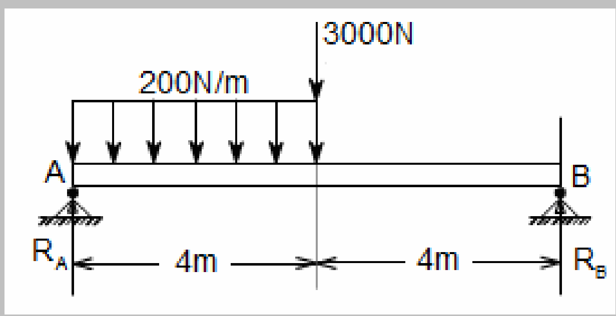
$$or \qquad d(M_x) = V_x dx = \left(\frac{wL}{2} - wx\right)dx$$

Integrating both side we get (at x =0, V_x =0)

$$\int_o^{M_x} d(M_x) = \int_0^x \left(\frac{wL}{2} - wx\right)dx$$

$$or\ M_x = \frac{wL}{2}.x - \frac{wx^2}{2}$$

Let us take an example: A loaded beam as shown below. Draw its S.F and B.M diagram.



Considering equilibrium we get

$$\sum M_A = 0 \text{ gives}$$

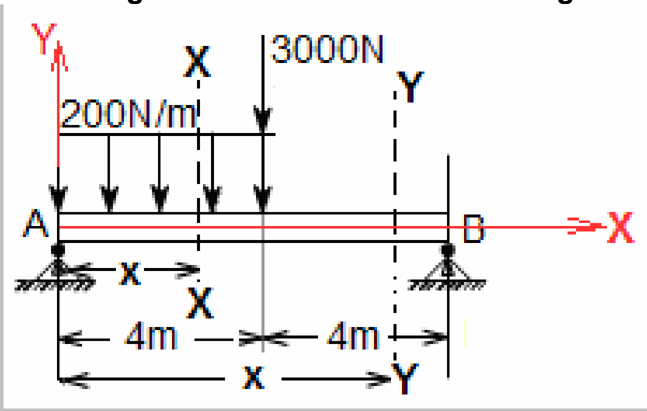
$$-(200 \times 4) \times 2 - 3000 \times 4 + R_B \times 8 = 0$$

$$or \qquad R_B = 1700\,N$$

$$And \quad R_A + R_B = 200 \times 4 + 3000$$

$$or \qquad R_A = 2100\,N$$

Now consider any cross-section XX which is at a distance 'x' from left end A and as shown in figure



In the region 0 < x < 4m

Shear force $(V_x) = R_A - 200x = 2100 - 200\,x$

Bending moment $(M_x) = R_A .x - 200\,x\,.\left(\dfrac{x}{2}\right) = 2100\,x - 100\,x^2$

at $x = 0$, $V_x = 2100$ N, $M_x = 0$

at $x = 4$m, $V_x = 1300$ N, $M_x = 6800$ N.m

In the region 4 m < x < 8 m

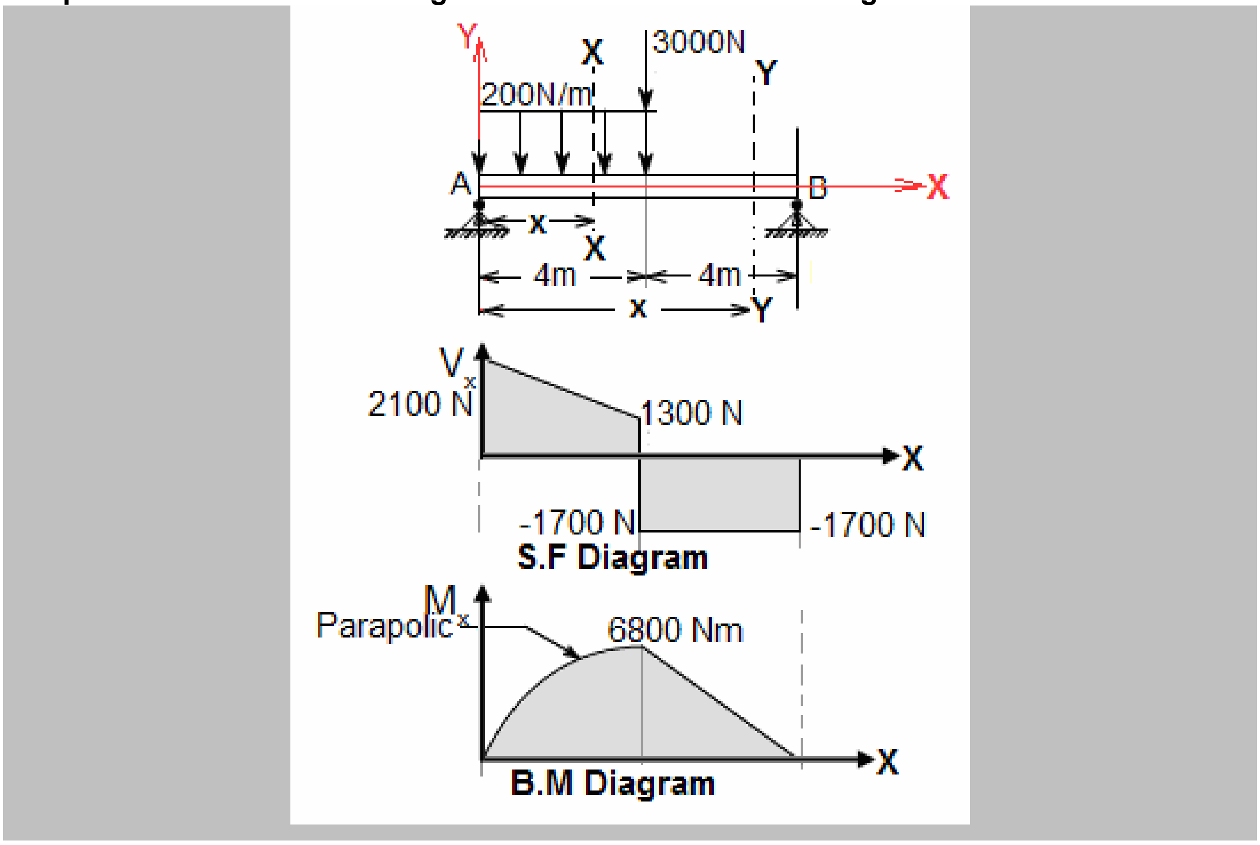
Shear force $(V_x) = R_A - 200 \times 4 - 3000 = -1700$

Bending moment $(M_x) = R_A.\,x - 200\times 4\,(x-2) - 3000\,(x-4)$

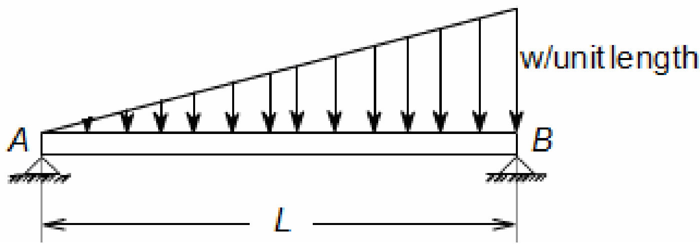
$= 2100\,x - 800\,x + 1600 - 3000x + 12000 = 13600 - 1700\,x$

at $x = 4$ m, $V_x = -1700$ N, $M_x = 6800$ Nm

at $x = 8$ m, $V_x = -1700$ N, $M_x = 0$



(xii) A Simply supported beam with a gradually varying load (GVL) zero at one end and w/unit length at other span.



Consider equilibrium of the beam $= \frac{1}{2}wL$ acting at a point C at a distance 2L/3 to the left end A.

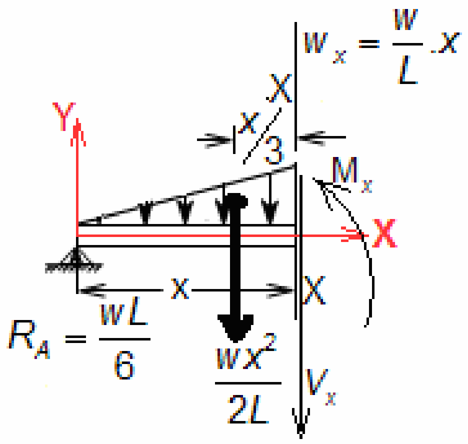
$$\sum M_B = 0 \text{ gives}$$

$$R_A.L - \frac{wL}{2}.\frac{L}{3} = 0$$

$$\text{or } R_A = \frac{wL}{6}$$

$$\text{Similarly } \sum M_A = 0 \text{ gives } R_B = \frac{wL}{3}$$

The free body diagram of section A - XX as shown below, Load at section XX, $(w_x) = \frac{w}{L}x$



The resulted of that part of the distributed load which acts on this free body is $=\frac{1}{2}(x).\frac{w}{L}x = \frac{wx^2}{2L}$

applied at a point Z, distance x/3 from XX section.

Shear force (V_x) $= R_A - \frac{wx^2}{2L} = \frac{wL}{6} - \frac{wx^2}{2L}$

Therefore the variation of shear force is parabolic

at $x = 0$, $V_x = \frac{wL}{6}$

at $x = L$, $V_x = -\frac{wL}{3}$

and **Bending Moment (M_x)** $= \frac{wL}{6}.x - \frac{wx^2}{2L}.\frac{x}{3} = \frac{wL}{6}.x - \frac{wx^3}{6L}$

The variation of BM is cubic

at $x = 0$, $M_x = 0$

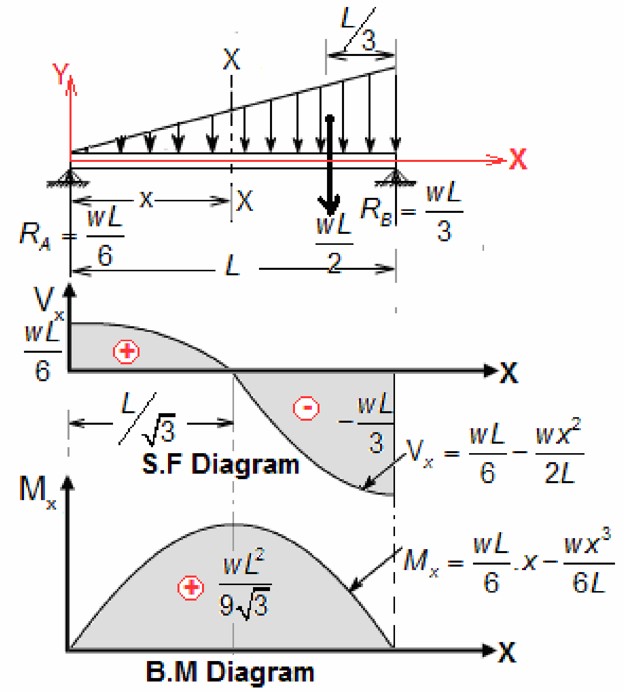
at $x = L$, $M_x = 0$

For maximum BM; $\frac{d(M_x)}{dx} = 0$ i.e. $V_x = 0$ $\left[\because \frac{d(M_x)}{dx} = V_x\right]$

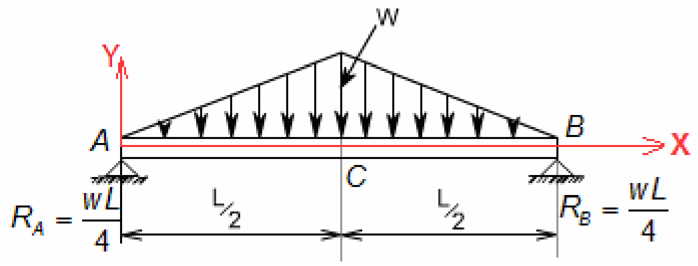
or $\frac{wL}{6} - \frac{wx^2}{2L} = 0$ or $x = \frac{L}{\sqrt{3}}$

and $M_{max} = \frac{wL}{6} \times \left(\frac{L}{\sqrt{3}}\right) - \frac{w}{6L} \times \left(\frac{L}{\sqrt{3}}\right)^3 = \frac{wL^2}{9\sqrt{3}}$

i.e. $$M_{max} = \frac{wL^2}{9\sqrt{3}}$$ at $x = \frac{L}{\sqrt{3}}$



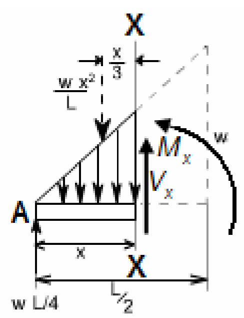
(xiii) A Simply supported beam with a gradually varying load (GVL) zero at each end and w/unit length at mid span.



Consider equilibrium of the beam AB total load on the beam $= 2 \times \left(\frac{1}{2} \times \frac{L}{2} \times w \right) = \frac{wL}{2}$

$$\text{Therefore } R_A = R_B = \frac{wL}{4}$$

The free body diagram of section A –XX as shown below, load at section XX (w_x) $= \frac{2w}{L}.x$



The resultant of that part of the distributed load which acts on this free body is $= \frac{1}{2}.x.\frac{2w}{L}.x = \frac{wx^2}{L}$

applied at a point, distance x/3 from section XX.

Shear force (V_x):

In the region 0 < x < L/2

$$\left(V_x\right) = R_A - \frac{wx^2}{L} = \frac{wL}{4} - \frac{wx^2}{L}$$

Therefore the variation of shear force is parabolic.

at x = 0, $V_x = \frac{wL}{4}$

at x = L/4, $V_x = 0$

In the region of L/2 < x < L

The Diagram will be Mirror image of AC.

Bending moment (M_x):

In the region 0 < x < L/2

$$M_x = \frac{wL}{4}.x - \left(\frac{1}{2}.x.\frac{2wx}{L}\right).\left(x/3\right) = \frac{wL}{4} - \frac{wx^3}{3L}$$

The variation of BM is cubic

at x = 0, $M_x = 0$

at x = L/2, $M_x = \frac{wL^2}{12}$

In the region L/2 < x < L

BM diagram will be mirror image of AC.

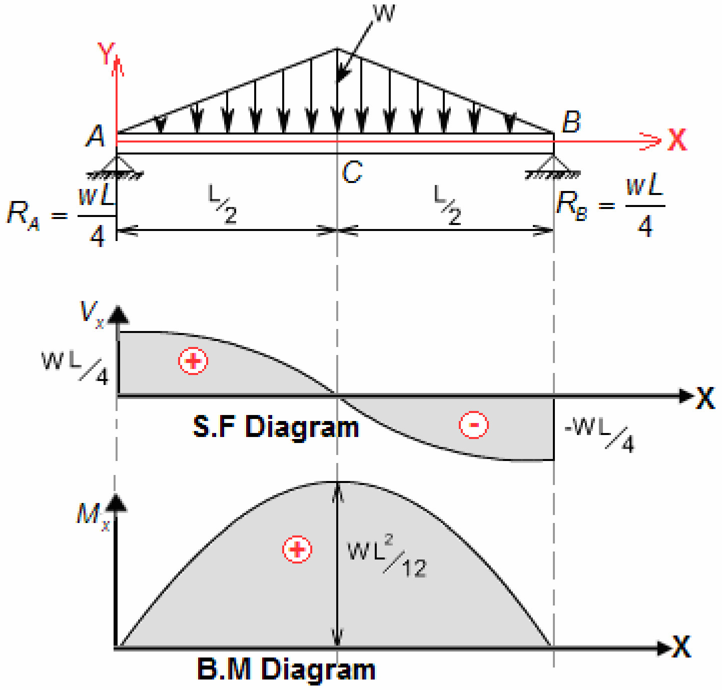
For maximum bending moment

$$\frac{d\left(M_x\right)}{dx} = 0 \quad \text{i.e. } V_x = 0 \qquad \left[\because \frac{d\left(M_x\right)}{dx} = V_x\right]$$

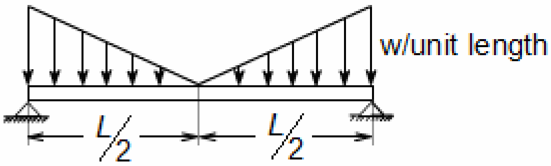
$$\text{or } \frac{wL}{4} - \frac{wx^2}{L} = 0 \quad \text{or} \quad x = \frac{L}{2}$$

$$\text{and } M_{max} = \frac{wL^2}{12}$$

i.e. $$\boxed{M_{max} = \frac{wL^2}{12}} \quad \text{at } x = \frac{L}{2}$$

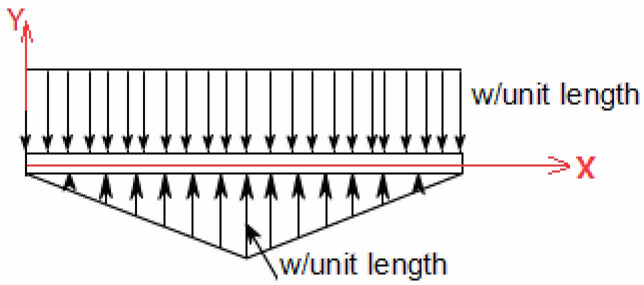


(xiv) A Simply supported beam with a gradually varying load (GVL) zero at mid span and w/unit length at each end.



We now superimpose two beams as

(1) Simply supported beam with a UDL through at its length



$$(V_x)_1 = \frac{wL}{2} - wx$$

$$(M_x)_1 = \frac{wL}{2}.x - \frac{wx^2}{2}$$

And (2) a simply supported beam with a gradually varying load (GVL) zero at each end and w/unit length at mind span.

In the range 0 < x < L/2

$$(V_x)_2 = \frac{wL}{4} - \frac{wx^2}{L}$$

$$(M_x)_2 = \frac{wL}{4}.x - \frac{wx^3}{3L}$$

Now superimposing we get

Shear force (V_x):

In the region of 0< x < L/2

$$V_x = (V_x)_1 - (V_x)_2 = \left(\frac{wL}{2} - wx\right) - \left(\frac{wL}{4} - \frac{wx^2}{L}\right)$$

$$= \frac{w}{L}(x - L/2)^2$$

Therefore the variation of shear force is parabolic

at $x = 0$, $\quad V_x = +\dfrac{wL}{4}$

at $x = L/2$, $\quad V_x = 0$

In the region L/2 < x < L

The diagram will be mirror image of AC

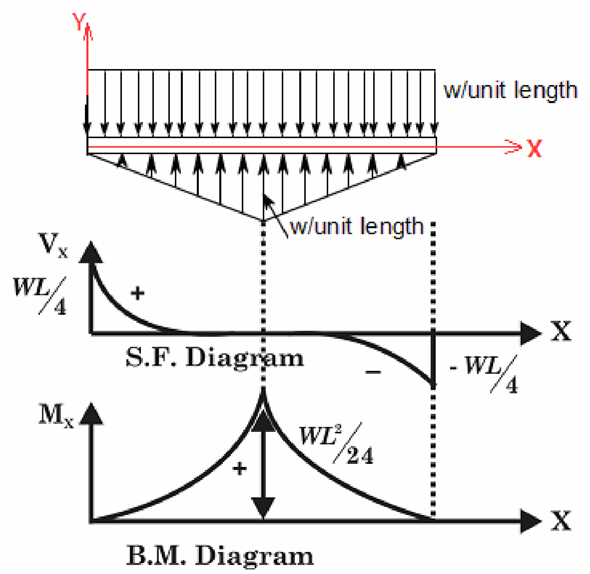
Bending moment (M_x) = $(M_x)_1 - (M_x)_2$ =

$$= \left(\frac{wL}{2}.x - \frac{wx^2}{2}\right) - \left(\frac{wL}{4}.x - \frac{wx^3}{3L}\right) = \frac{wx^3}{3L} - \frac{wx^2}{2} + \frac{wL}{4}.x$$

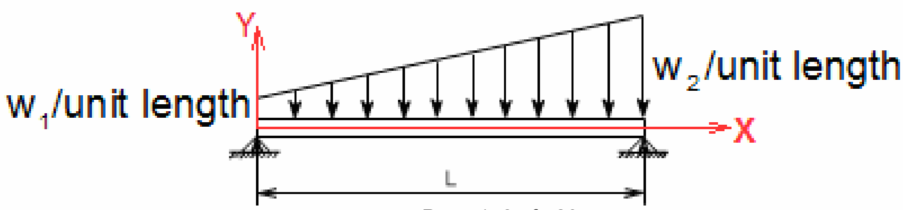
The variation of BM is cubic

$$\text{at } x = 0,\ M_x = 0$$

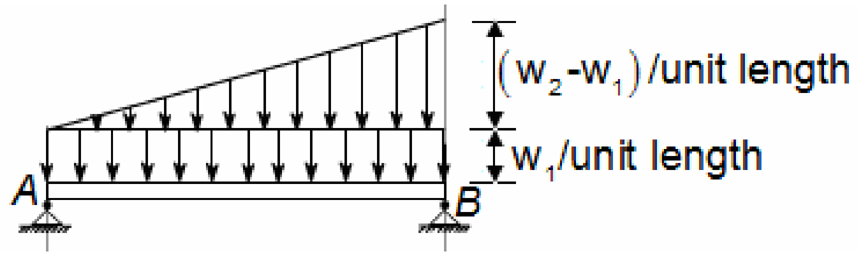
$$\text{at } x = L/2,\ M_x = \frac{wx^2}{24}$$



(xv) A simply supported beam with a gradually varying load (GVL) w_1/unit length at one end and w_2/unit length at other end.



At first we will treat this problem by considering a UDL of identifying (w_1)/unit length over the whole length and a varying load of zero at one end to (w_2- w_1)/unit length at the other end. Then superimpose the two loadings.



Consider a section XX at a distance x from left end A

(i) Simply supported beam with UDL (w_1) over whole length

$$(V_x)_1 = \frac{w_1L}{2} - w_1x$$

$$(M_x)_1 = \frac{w_1L}{2}.x - \frac{1}{2}w_1x^2$$

And (ii) simply supported beam with (GVL) zero at one end (w_2- w_1) at other end gives

$$(V_x)_2 = \frac{(w_2 - w_1)}{6} - \frac{(w_2 - w_1)x^2}{2L}$$

$$(M_x)_2 = (w_2 - w_1).\frac{L}{6}.x - \frac{(w_2 - w_1)x^3}{6L}$$

Now superimposing we get

Shear force $(\mathbf{V_x}) = (V_x)_1 + (V_x)_2 = \frac{w_1L}{3} + \frac{w_2L}{6} - w_1x - (w_2 - w_1)\frac{x^2}{2L}$

$\therefore$ The SF variation is parabolic

$$\text{at } x = 0, \quad V_x = \frac{w_1L}{3} + \frac{w_2L}{6} = \frac{L}{6}(2w_1 + w_2)$$

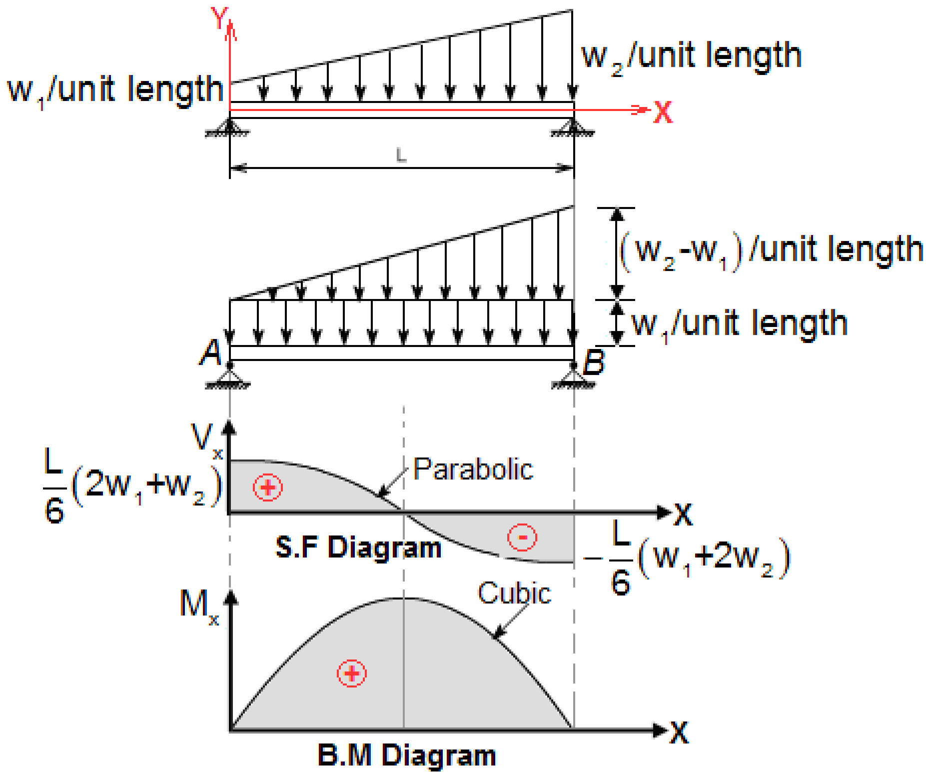
$$\text{at } x = L, \quad V_x = -\frac{L}{6}(w_1 + 2w_2)$$

Bending moment $(\mathbf{M_x}) = (M_x)_1 + (M_x)_2 = \frac{w_1L}{3}.x + \frac{w_1L}{6}.x - \frac{1}{2}w_1x^2 - \left(\frac{w_2\text{-}w_1}{6L}\right).x^3$

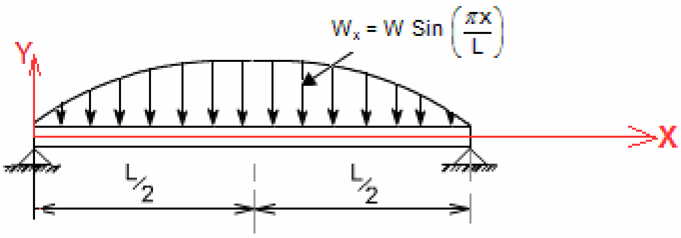
$\therefore$ The BM variation is cubic.

$$\text{at } x = 0, \quad M_x = 0$$

$$\text{at } x = L, \quad M_x = 0$$



(xvi) A Simply supported beam carrying a continuously distributed load. The intensity of the load at any point is, $w_x = w\sin\left(\frac{\pi x}{L}\right)$. Where 'x' is the distance from each end of the beam.



We will use Integration method as it is easier in this case.

We know that $\frac{d(V_x)}{dx} = \text{load}$ and $\frac{d(M_x)}{dx} = V_x$

Therefore $\frac{d(V_x)}{dx} = -w\sin\left(\frac{\pi x}{L}\right)$

$$d(V_x) = -w\sin\left(\frac{\pi x}{L}\right)dx$$

Integrating both side we get

$$\int d(V_x) = -w\int \sin\left(\frac{\pi x}{L}\right)dx \quad \text{or} \quad V_x = +\frac{w\cos\left(\frac{\pi x}{L}\right)}{\frac{\pi}{L}} + A = +\frac{wL}{\pi}\cos\left(\frac{\pi x}{L}\right) + A$$

[where, A = constant of Integration]

Again we know that

$$\frac{d(M_x)}{dx} = V_x \qquad \text{or} \qquad d(M_x) = V_x\, dx = \left\{\frac{wL}{\pi}\cos\left(\frac{\pi x}{L}\right) + A\right\}dx$$

Integrating both side we get

$$M_x = \frac{\frac{wL}{\pi}\sin\left(\frac{\pi x}{L}\right)}{\frac{\pi}{L}} + Ax + B = \frac{wL^2}{\pi^2}\sin\left(\frac{\pi x}{L}\right) + Ax + B$$

[Where B = constant of Integration]

Now apply boundary conditions

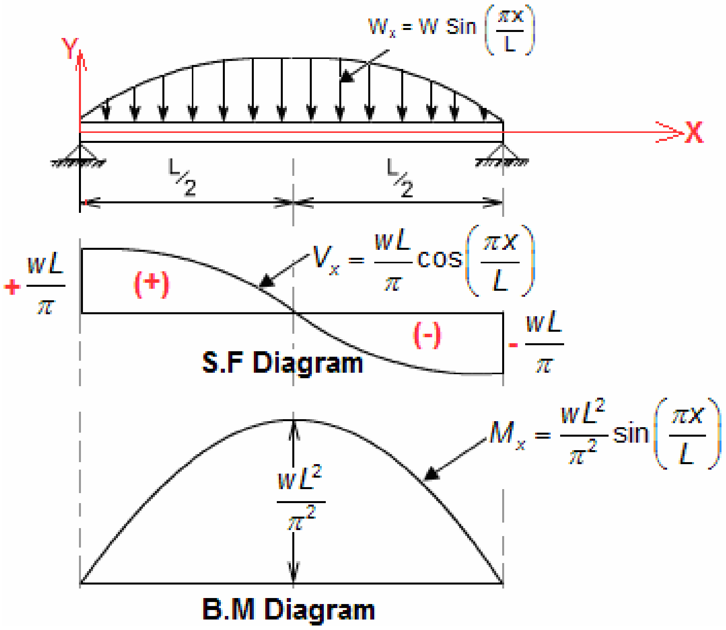
At $x = 0$, $M_x = 0$ and at $x = L$, $M_x = 0$

This gives A = 0 and B = 0

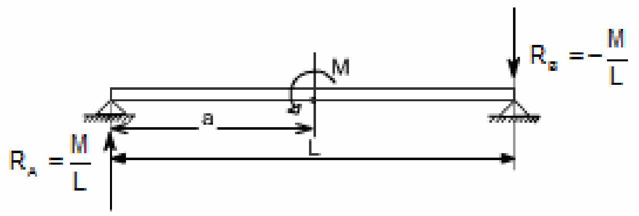
$$\therefore \text{ Shear force } (V_x) = \frac{wL}{\pi}\cos\left(\frac{\pi x}{L}\right) \qquad \text{and} \qquad V_{max} = \frac{wL}{\pi} \text{ at } x = 0$$

$$\text{And } M_x = \frac{wL^2}{\pi^2}\sin\left(\frac{\pi x}{L}\right)$$

$$\therefore \quad M_{max} = \frac{wL^2}{\pi^2} \qquad \text{at } x = L/2$$



(xvii) A Simply supported beam with a couple or moment at a distance 'a' from left end.



Considering equilibrium we get

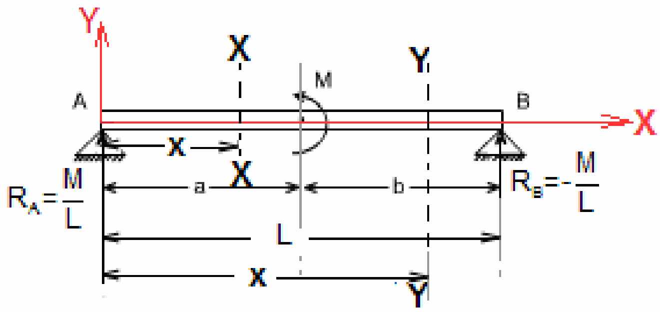
$\sum M_A = 0$ gives

$$R_B \times L + M = 0 \quad or \;\; R_B = -\frac{M}{L}$$

and $\sum M_B = 0$ gives

$$-R_A \times L + M = 0 \quad or \; R_A = \frac{M}{L}$$

Now consider any cross-section XX which is at a distance 'x' from left end A and another section YY at a distance 'x' from left end A as shown in figure.



In the region 0 < x < a

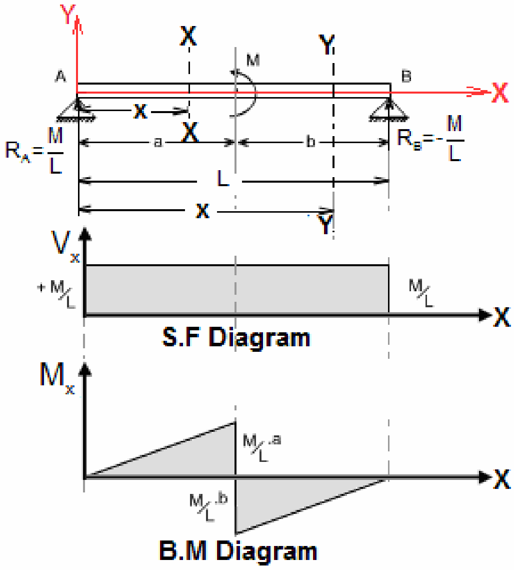
Shear force $(V_x) = R_A = \frac{M}{L}$

Bending moment $(M_x) = R_A.x = \frac{M}{L}.x$

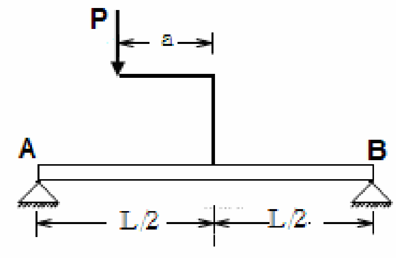
In the region a< x < L

Shear force $(V_x) = R_A = \frac{M}{L}$

Bending moment $(M_x) = R_A.x - M = \frac{M}{L}.x - M$



(xviii) A Simply supported beam with an eccentric load

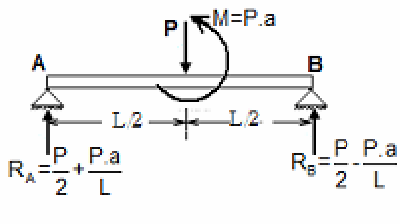


Actual loaded beam

When the beam is subjected to an eccentric load, the eccentric load is to be changed into a couple = Force $\times$ (distance travel by force)

= P.a (in this case) and a force = P

Therefore equivalent load diagram will be



Equivalent loaded beam

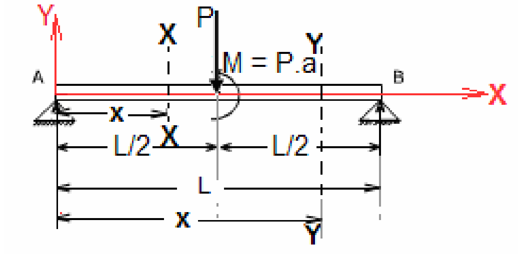
Considering equilibrium

$$\sum M_A = 0 \text{ gives}$$

$-P.(L/2) + P.a + R_B \times L = 0$

or $R_B = \frac{P}{2} - \frac{P.a}{L}$ and $R_A + R_B = P$ gives $R_A = \frac{P}{2} + \frac{P.a}{L}$

Now consider any cross-section XX which is at a distance 'x' from left end A and another section YY at a distance 'x' from left end A as shown in figure.



In the region 0 < x < L/2

$$\text{Shear force } (V_x) = \frac{P}{2} + \frac{P.a}{L}$$

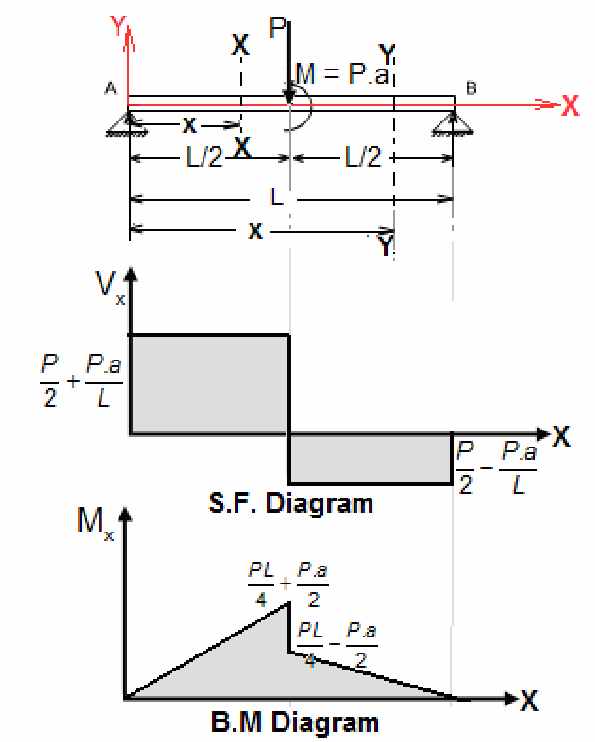
$$\text{Bending moment } (M_x) = R_A \cdot x = \left(\frac{P}{2} + \frac{Pa}{L}\right) \cdot x$$

In the region L/2 < x < L

$$\text{Shear force } (V_x) = \frac{P}{2} + \frac{Pa}{L} - P \quad = - \frac{P}{2} + \frac{Pa}{L}$$

$$\text{Bending moment } (V_x) = R_A \cdot x - P.(x - L/2) - M$$

$$= \frac{PL}{2} - \left(\frac{P}{2} - \frac{Pa}{L}\right).x - Pa$$

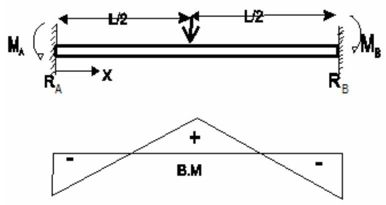
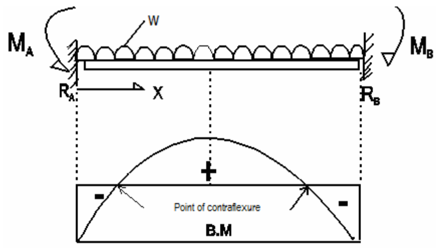
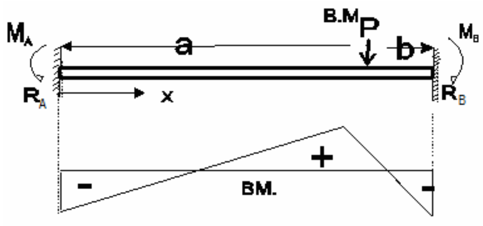
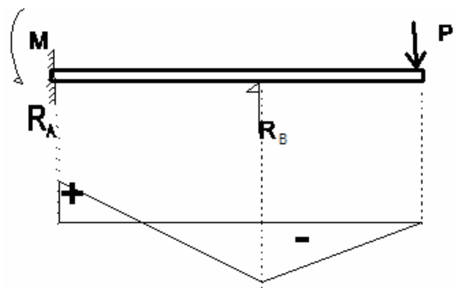
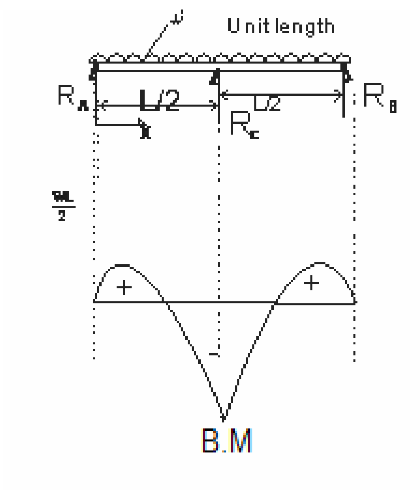
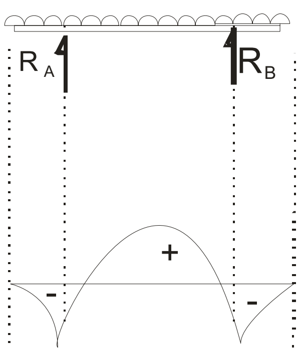


4.6 Bending Moment diagram of Statically Indeterminate beam

Beams for which reaction forces and internal forces **cannot** be found out from static equilibrium equations alone are called statically indeterminate beam. This type of beam requires deformation equation in addition to static equilibrium equations to solve for unknown forces.

Statically determinate - Equilibrium conditions sufficient to compute reactions.

Statically indeterminate - Deflections (Compatibility conditions) along with equilibrium equations should be used to find out reactions.

Type of Loading & B.M Diagram	Reaction	Bending Moment
	$R_A = R_B = \dfrac{P}{2}$	$M_A = M_B = -\dfrac{PL}{8}$
	$R_A = R_B = \dfrac{wL}{2}$	$M_A = M_B = -\dfrac{wL^2}{12}$
	$R_A = \dfrac{Pb^2}{L^3}(3a+b)$ $R_B = \dfrac{Pa^2}{L^3}(3b+a)$	$M_A = -\dfrac{Pab^2}{L^2}$ $M_B = -\dfrac{Pa^2b}{L^2}$
		
	$R_A = R_B = \dfrac{3wL}{16}$ $R_c = \dfrac{5wL}{8}$	
		

4.7 Load and Bending Moment diagram from Shear Force diagram OR Load and Shear Force diagram from Bending Moment diagram

If S.F. Diagram for a beam is given, then

(i) If S.F. diagram consists of rectangle then the load will be point load

(ii) If S.F diagram consists of inclined line then the load will be UDL on that portion

(iii) If S.F diagram consists of parabolic curve then the load will be GVL

(iv) If S.F diagram consists of cubic curve then the load distribute is parabolic.

After finding load diagram we can draw B.M diagram easily.

If B.M Diagram for a beam is given, then

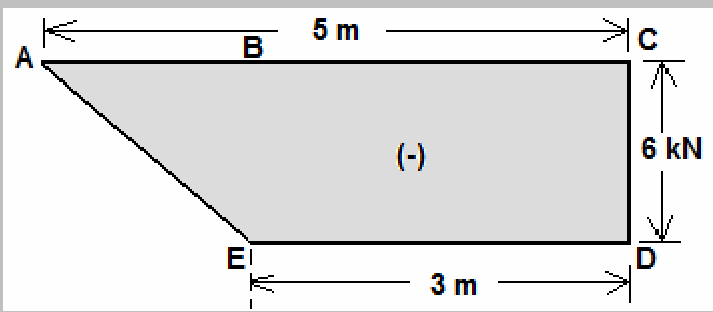
(i) If B.M diagram consists of inclined line then the load will be free point load

(ii) If B.M diagram consists of parabolic curve then the load will be U.D.L.

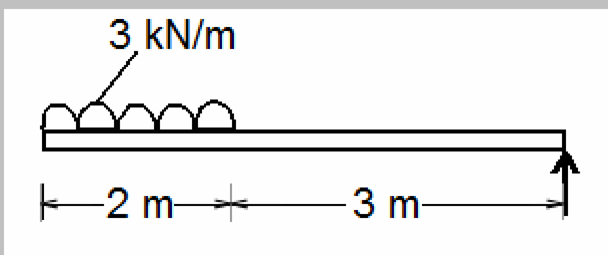
(iii) If B.M diagram consists of cubic curve then the load will be G.V.L.

(iv) If B.M diagram consists of fourth degree polynomial then the load distribution is parabolic.

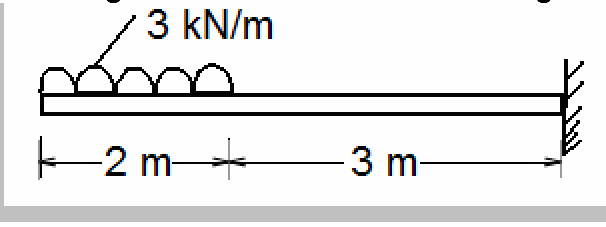
Let us take an example: Following is the S.F diagram of a beam is given. Find its loading diagram.



Answer: From A-E inclined straight line so load will be UDL and in AB = 2 m length load = 6 kN if UDL is w N/m then $w.x = 6$ or $w \times 2 = 6$ or $w = 3$ kN/m after that S.F is constant so no force is there. At last a 6 kN for vertical force complete the diagram then the load diagram will be



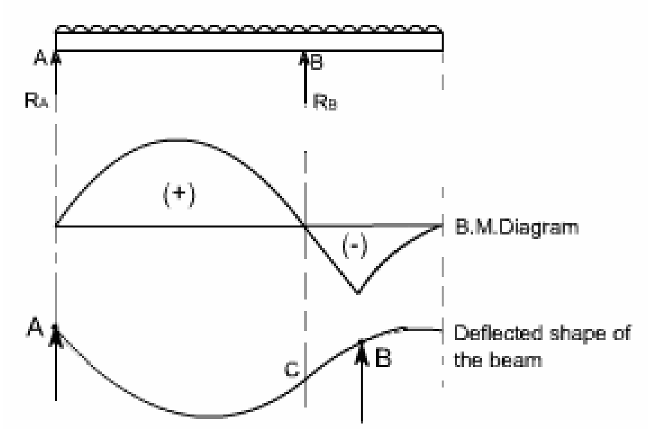
As there is no support at left end it must be a cantilever beam.



4.8 Point of Contraflexure

In a beam if the bending moment changes sign at a point, the point itself having zero bending moment, the beam changes curvature at this point of zero bending moment and this point is called the point of contra flexure.

Consider a loaded beam as shown below along with the B.M diagrams and deflection diagram.



In this diagram we noticed that for the beam loaded as in this case, the bending moment diagram is partly positive and partly negative. In the deflected shape of the beam just below the bending moment diagram shows that left hand side of the beam is 'sagging' while the right hand side of the beam is 'hogging'.

The point C on the beam where the curvature changes from sagging to hogging is a point of contraflexure.

- ***There can be more than one point of contraflexure in a beam.***

4.9 General expression

- $\mathrm{EI}\dfrac{d^4y}{dx^2} = -\omega$
- $EI\dfrac{d^3y}{dx^3} = V_x$
- $EI\dfrac{d^2y}{dx^2} = M_x$

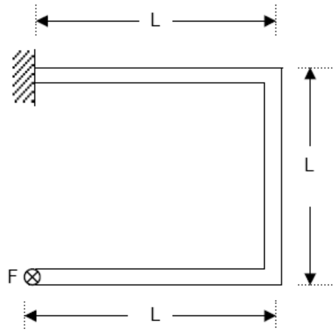
- $\frac{dy}{dx} = \theta = \text{slope}$
- $y = \delta$ = Deflection
- Flexural rigidity = EI

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Shear Force (S.F.) and Bending Moment (B.M.)

GATE-1. A concentrated force, F is applied (perpendicular to the plane of the figure) on the tip of the bent bar shown in Figure. The equivalent load at a section close to the fixed end is:



(a) Force F
(b) Force F and bending moment FL
(c) Force F and twisting moment FL
(d) Force F bending moment F L, and twisting moment FL

[GATE-1999]

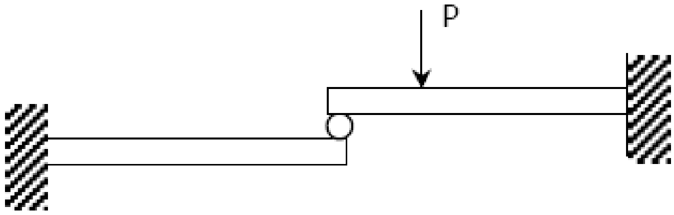
GATE-1. Ans. (c)

GATE-2. The shear force in a beam subjected to pure positive bending is...... (positive/zero/negative) [GATE-1995]

GATE-2. Ans. Zero

Cantilever

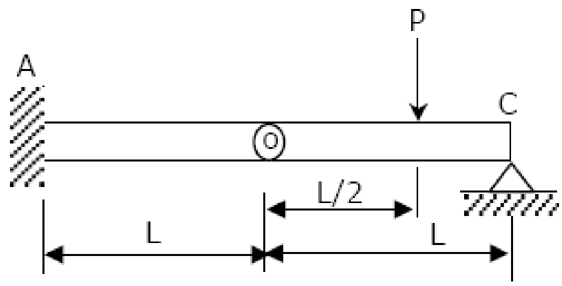
GATE-3. Two identical cantilever beams are supported as shown, with their free ends in contact through a rigid roller. After the load P is applied, the free ends will have [GATE-2005]



(a) Equal deflections but not equal slopes
(b) Equal slopes but not equal deflections
(c) Equal slopes as well as equal deflections
(d) Neither equal slopes nor equal deflections

GATE-3. Ans. (a) As it is rigid roller, deflection must be same, because after deflection they also will be in contact. But slope unequal.

GATE-4. A beam is made up of two identical bars AB and BC, by hinging them together at B. The end A is built-in (cantilevered) and the end C is simply-supported. With the load P acting as shown, the bending moment at A is:



[GATE-2005]

(a) Zero (b) $\dfrac{PL}{2}$ (c) $\dfrac{3PL}{2}$ (d) Indeterminate

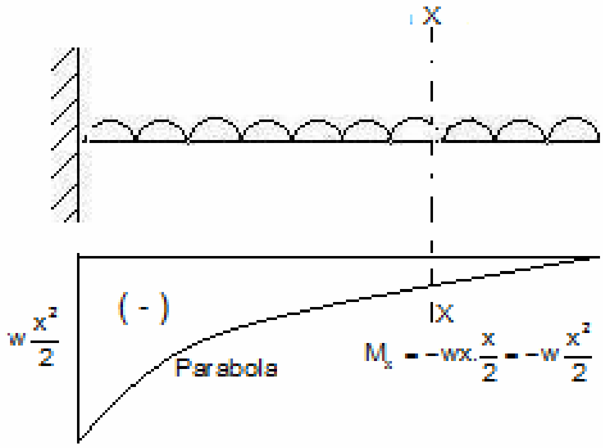
GATE-4. Ans. (b)

Cantilever with Uniformly Distributed Load

GATE-5. The shapes of the bending moment diagram for a uniform cantilever beam carrying a uniformly distributed load over its length is: [GATE-2001]

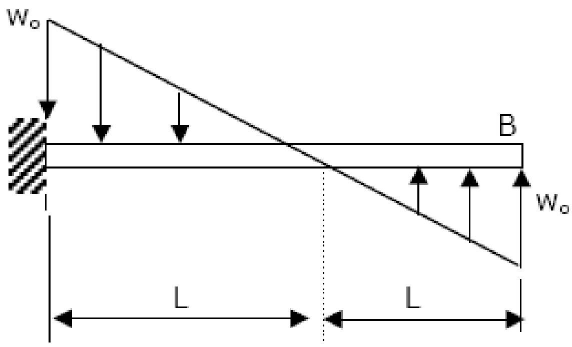
(a) A straight line (b) A hyperbola (c) An ellipse (d) A parabola

GATE-5. Ans. (d)

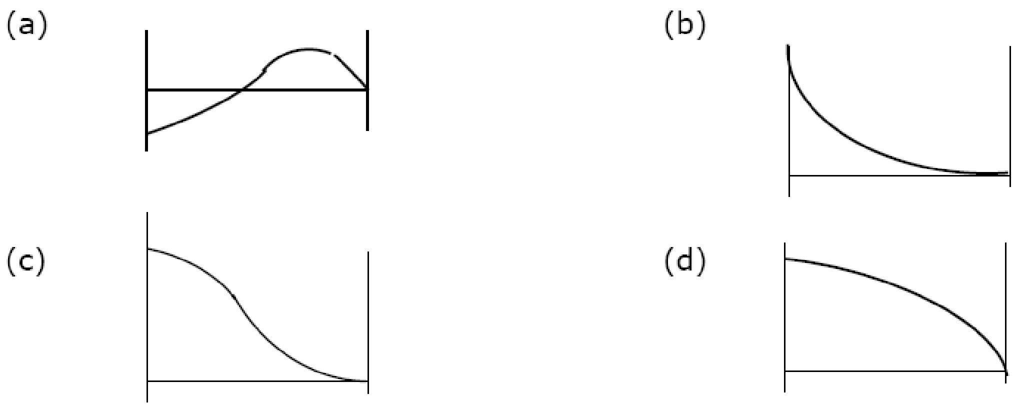


Cantilever Carrying load Whose Intensity varies

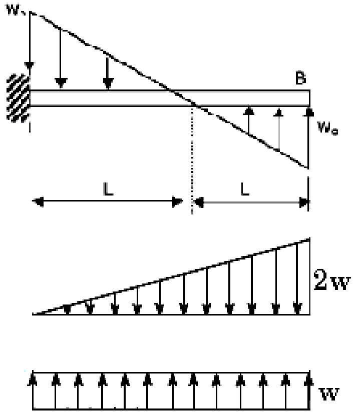
GATE-6. A cantilever beam carries the anti-symmetric load shown, where ω_o is the peak intensity of the distributed load. Qualitatively, the correct bending moment diagram for this beam is:



[GATE-2005]



GATE-6. Ans. (d)



$$M_x = \frac{wx^2}{2} - \frac{wx^3}{6L}$$

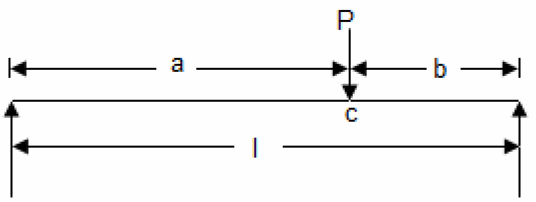
Simply Supported Beam Carrying Concentrated Load

GATE-7. A concentrated load of P acts on a simply supported beam of span L at a distance $\frac{L}{3}$ from the left support. The bending moment at the point of application of the load is given by [GATE-2003]

$(a)\frac{PL}{3}$ $(b)\frac{2PL}{3}$ $(c)\frac{PL}{9}$ $(d)\frac{2PL}{9}$

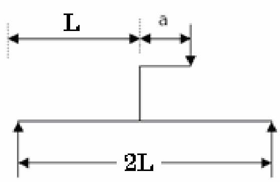
GATE-7. Ans. (d)

$$M_c = \frac{Pab}{l} = \frac{P\times\left(\frac{L}{3}\right)\times\left(\frac{2L}{3}\right)}{L} = \frac{2PL}{9}$$



GATE-8. A simply supported beam carries a load 'P' through a bracket, as shown in Figure. The maximum bending moment in the beam is

(a) Pl/2 (b) Pl/2 + aP/2
(c) Pl/2 + aP (d) Pl/2 – aP



[GATE-2000]

GATE-8. Ans. (c)

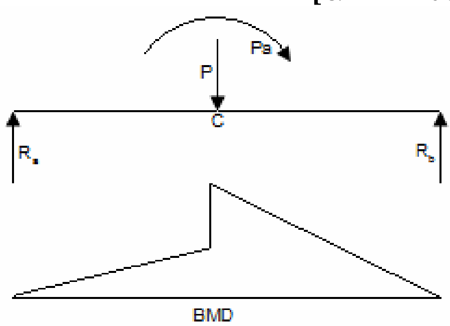
Taking moment about R_a

$$-P\times l/2 - Pa + R_b l = 0$$

$$\text{or } R_b = \frac{P}{2} + P\frac{a}{l} \quad \therefore R_a = \frac{P}{2} - P\frac{a}{l}$$

Maximum bending moment will be at centre 'C'

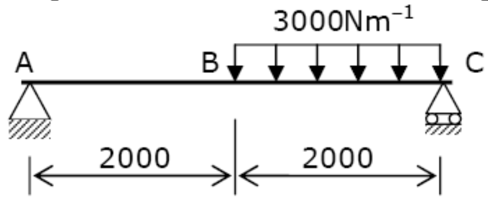
$$\therefore M_c = R_a\times\frac{l}{2} + P\times a + R_b\times\frac{l}{2} \quad \text{or } M_{max} = \frac{Pl}{2} + Pa$$



Simply Supported Beam Carrying a Uniformly Distributed Load

Statement for Linked Answer and Questions Q9-Q10:

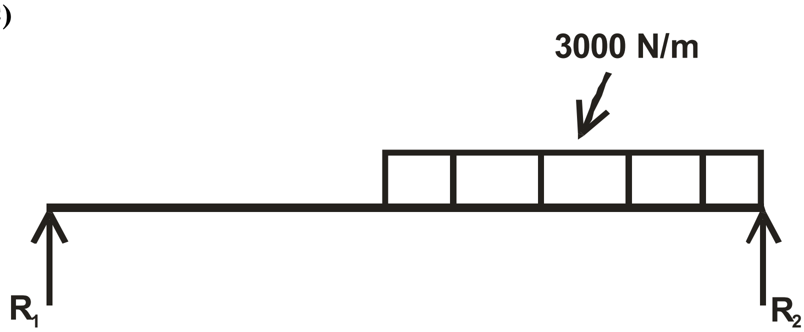
A mass less beam has a loading pattern as shown in the figure. The beam is of rectangular cross-section with a width of 30 mm and height of 100 mm. [GATE-2010]



GATE-9. The maximum bending moment occurs at

(a) Location B (b) 2675 mm to the right of A
(c) 2500 mm to the right of A (d) 3225 mm to the right of A

GATE-9. Ans. (C)



$R_1 + R_2 = 3000 \times 2 = 6000\,\text{N}$

$R_1 \times 4 - 3000 \times 2 \times 1 = 0$

$R_1 = 1500,$

S.F. eq^n. at any section x from end A.

$R_1 - 3000 \times (x - 2) = 0 \qquad \{\text{for} \quad x > 2\text{m}\}$

$x = 2.5$ m.

GATE-10. The maximum magnitude of bending stress (in MPa) is given by

(a) 60.0 (b) 67.5 (c) 200.0 (d) 225.0

GATE-10. Ans. (b)

Binding stress will be maximum at the outer surface
So taking 9 = 50 mm

$$\text{and } I = \frac{ld^3}{12} \quad \& \quad \sigma = \frac{m \times 50}{l\,d^3/12}$$

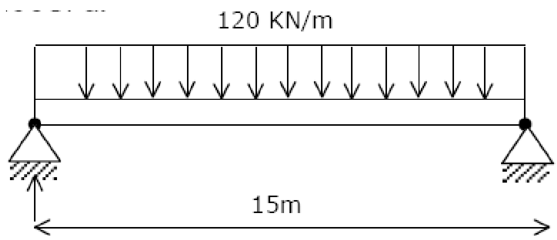
$$m_x = 1.5 \times 10^3[2000 + x] - \frac{x^2}{2}$$

$$\therefore \; m_{2500} = 3.375 \times 10^6 \; N - mm$$

$$\therefore \; \sigma = \frac{3.375 \times 10^6 \times 50 \times 12}{30 \times 100^3} = 67.5\,MPa$$

Data for Q11-Q12 are given below. Solve the problems and choose correct answers

A steel beam of breadth 120 mm and height 750 mm is loaded as shown in the figure. Assume E_{steel}= 200 GPa.



[GATE-2004]

GATE-11. The beam is subjected to a maximum bending moment of

(a) 3375 kNm (b) 4750 kNm (c) 6750 kNm (d) 8750 kNm

GATE-11. Ans. (a) $M_{max} = \frac{wl^2}{8} = \frac{120 \times 15^2}{8} kNm = 3375 kNm$

GATE-12. The value of maximum deflection of the beam is:
(a) 93.75 mm (b) 83.75 mm (c) 73.75 mm (d) 63.75 mm

GATE-12. Ans. (a) Moment of inertia (I) = $\frac{bh^3}{12} = \frac{0.12 \times (0.75)^3}{12} = 4.22 \times 10^{-3} m^4$

$$\delta_{max} = \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{384} \times \frac{120 \times 10^3 \times 15^4}{200 \times 10^9 \times 4.22 \times 10^{-3}} m = 93.75 mm$$

Statement for Linked Answer and Questions Q13-Q14:

A simply supported beam of span length 6m and 75mm diameter carries a uniformly distributed load of 1.5 kN/m [GATE-2006]

GATE-13. What is the maximum value of bending moment?
(a) 9 kNm (b) 13.5 kNm (c) 81 kNm (d) 125 kNm

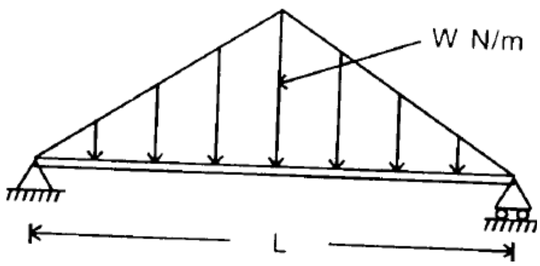
GATE-13. Ans. (a) $M_{max} = \frac{wl^2}{8} = \frac{1.5 \times 6^2}{8} = 6.75 kNm$ But not in choice. Nearest choice (a)

GATE-14. What is the maximum value of bending stress?
(a) 162.98 MPa (b) 325.95 MPa (c) 625.95 Mpa (d) 651.90 Mpa

GATE-14. Ans. (a) $\sigma = \frac{32M}{\pi d^3} = \frac{32 \times 6.75 \times 10^3}{\pi \times (0.075)^2} Pa = 162.98 MPa$

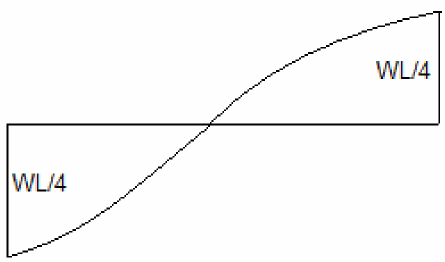
Simply Supported Beam Carrying a Load whose Intensity varies Uniformly from Zero at each End to w per Unit Run at the MiD Span

GATE-15. A simply supported beam is subjected to a distributed loading as shown in the diagram given below:
What is the maximum shear force in the beam?
(a) WL/3 (b) WL/2
(c) WL/3 (d) WL/6



[IES-2004]

GATE-15. Ans. (d)



$$\text{Total load} = \frac{1}{2} \times L \times W = \frac{WL}{2}$$

$$S_x = \frac{WL}{4} - \frac{1}{2} x. \left(\frac{W}{\frac{L}{2}} \times X \right) = \frac{WL}{4} - \frac{Wx^2}{L}$$

$$S_{max \text{ at } x=0} = \frac{WL}{4}$$

GATE-16. A simply supported beam of length 'l' is subjected to a symmetrical uniformly varying load with zero intensity at the ends and intensity w (load per unit length) at the mid span. What is the maximum bending moment? [IAS-2004]

(a) $\dfrac{3wl^2}{8}$ (b) $\dfrac{wl^2}{12}$ (c) $\dfrac{wl^2}{24}$ (d) $\dfrac{5wl^2}{12}$

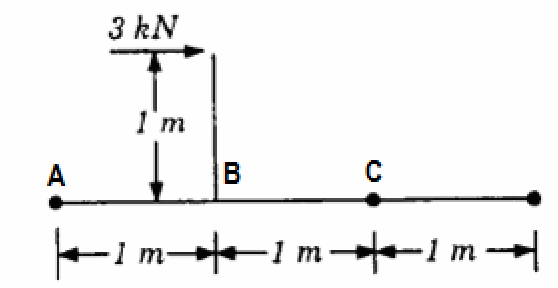
GATE-16. Ans. (b)

Previous 20-Years IES Questions

Shear Force (S.F.) and Bending Moment (B.M.)

IES-1. A lever is supported on two hinges at A and C. It carries a force of 3 kN as shown in the above figure. The bending moment at B will be

(a) 3 kN-m (b) 2 kN-m
(c) 1 kN-m (d) Zero

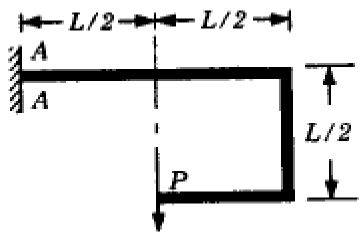


[IES-1998]

IES-1. Ans. (a)

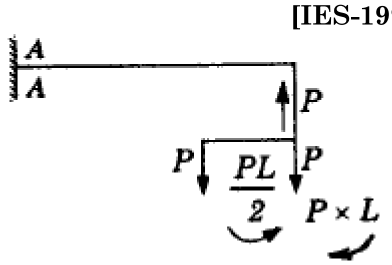
IES-2. A beam subjected to a load P is shown in the given figure. The bending moment at the support AA of the beam will be

(a) PL (b) PL/2
(c) 2PL (d) zero



[IES-1997]

IES-2. Ans. (b) Load P at end produces moment $\dfrac{PL}{2}$ in anticlockwise direction. Load P at end produces moment of PL in clockwise direction. Net moment at AA is PL/2.



IES-3. The bending moment (M) is constant over a length segment (I) of a beam. The shearing force will also be constant over this length and is given by [IES-1996]

(a) M/l (b) M/2l (c) M/4l (d) None of the above

IES-3. Ans. (d) Dimensional analysis gives choice (d)

IES-4. A rectangular section beam subjected to a bending moment M varying along its length is required to develop same maximum bending stress at any cross-section. If the depth of the section is constant, then its width will vary as [IES-1995]

(a) M (b) $\sqrt{M}$ (c) M^2 (d) 1/M

IES-4. Ans. (a) $\dfrac{M}{I} = \text{const.}$ and $I = \dfrac{bh^3}{12}$

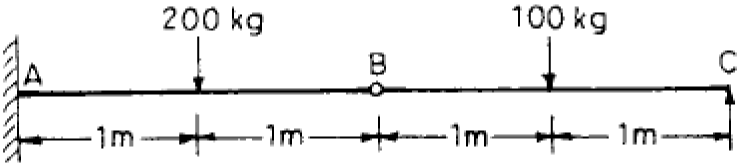
IES-5. Consider the following statements: [IES-1995]
If at a section distant from one of the ends of the beam, M represents the bending moment. V the shear force and w the intensity of loading, then
1. dM/dx = V 2. dV/dx = w
3. dw/dx = y (the deflection of the beam at the section)
Select the correct answer using the codes given below:

(a) 1 and 3 (b) 1 and 2 (c) 2 and 3 (d) 1, 2 and 3

IES-5. Ans. (b)

Cantilever

IES-6. The given figure shows a beam BC simply supported at C and hinged at B (free end) of a cantilever AB. The beam and the cantilever carry forces of



100 kg and 200 kg respectively. The bending moment at B is: [IES-1995]

(a) Zero (b) 100 kg-m (c) 150 kg-m (d) 200 kg-m

IES-6. Ans. (a)

IES-7. Match List-I with List-II and select the correct answer using the codes given below the lists: [IES-1993]

List-I (Condition of beam)	**List-II (Bending moment diagram)**
A. Subjected to bending moment at the end of a cantilever	**1.** Triangle
B. Cantilever carrying uniformly distributed load over the whole length	**2.** Cubic parabola
C. Cantilever carrying linearly varying load from zero at the fixed end to maximum at the support	**3.** Parabola
D. A beam having load at the centre and supported at the ends	**4.** Rectangle

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	4	1	2	3	(b)	4	3	2	1
(c)	3	4	2	1	(d)	3	4	1	2

IES-7. Ans. (b)

IES-8. If the shear force acting at every section of a beam is of the same magnitude and of the same direction then it represents a [IES-1996]

(a) Simply supported beam with a concentrated load at the centre.
(b) Overhung beam having equal overhang at both supports and carrying equal concentrated loads acting in the same direction at the free ends.
(c) Cantilever subjected to concentrated load at the free end.
(d) Simply supported beam having concentrated loads of equal magnitude and in the same direction acting at equal distances from the supports.

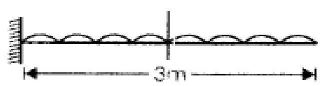
IES-8. Ans. (c)

Cantilever with Uniformly Distributed Load

IES-9. A uniformly distributed load ω (in kN/m) is acting over the entire length of a 3 m long cantilever beam. If the shear force at the midpoint of cantilever is 6 kN, what is the value of ω? [IES-2009]

(a) 2 (b) 3 (c) 4 (d) 5

IES-9. Ans. (c)



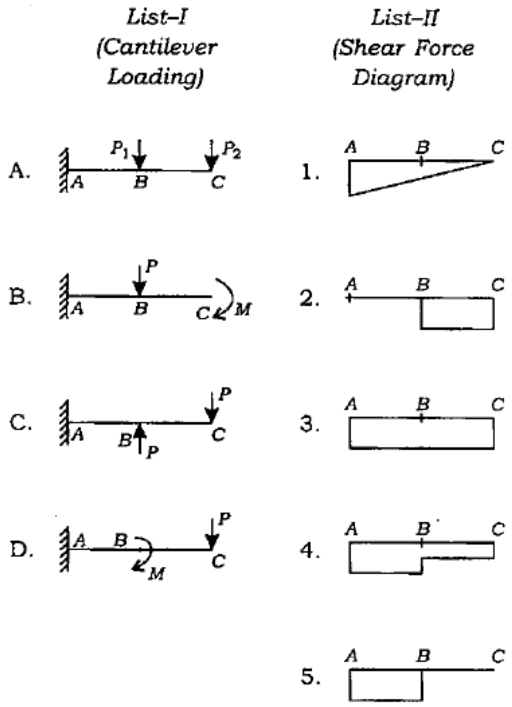
Shear force at mid point of cantilever

$$= \frac{\omega l}{2} = 6$$

$$\Rightarrow \quad \frac{\omega \times 3}{2} = 6$$

$$\Rightarrow \quad \omega = \frac{6 \times 2}{3} = 4 \text{ kN/m}$$

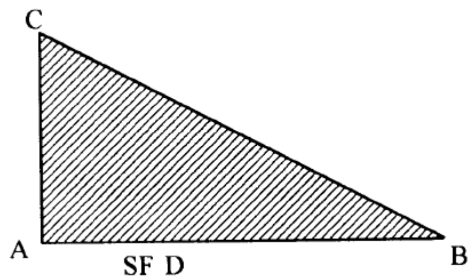
IES-10. **Match List-I with List-II and select the correct answer using the code given below the Lists:** **[IES-2009]**



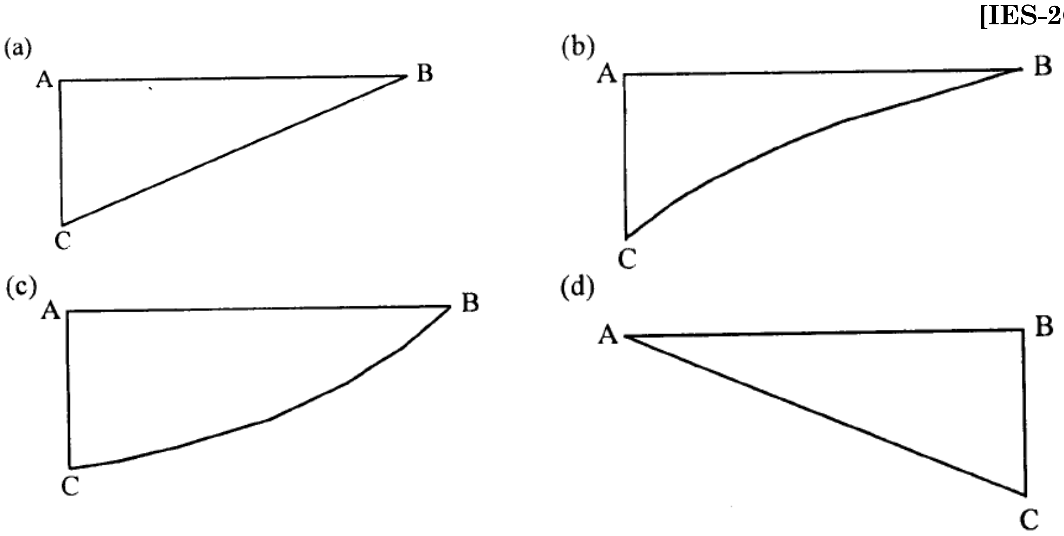
Code:	A	B	C	D		A	B	C	D
(a)	1	5	2	4	(b)	4	5	2	3
(c)	1	3	4	5	(d)	4	2	5	3

IES-10. Ans. (b)

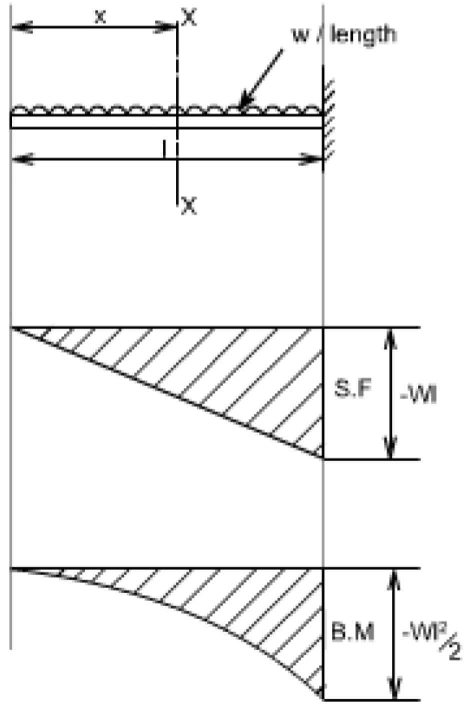
IES-11. **The shearing force diagram for a beam is shown in the above figure. The bending moment diagram is represented by which one of the following?**



[IES-2008]



IES-11. Ans. (b) Uniformly distributed load on cantilever beam.



IES-12. **A cantilever beam having 5 m length is so loaded that it develops a shearing force of 20T and a bending moment of 20 T-m at a section 2m from the free end. Maximum shearing force and maximum bending moment developed in the beam under this load are respectively 50 T and 125 T-m. The load on the beam is: [IES-1995]**

(a) 25 T concentrated load at free end
(b) 20T concentrated load at free end
(c) 5T concentrated load at free end and 2 T/m load over entire length
(d) 10 T/m udl over entire length

IES-12. Ans. (d)

Cantilever Carrying Uniformly Distributed Load for a Part of its Length

IES-13. **A vertical hanging bar of length L and weighing w N/ unit length carries a load W at the bottom. The tensile force in the bar at a distance Y from the support will be given by [IES-1992]**

(a) $W + wL$ (b) $W + w(L - y)$ (c) $(W + w)y / L$ (d) $W + \frac{W}{w}(L - y)$

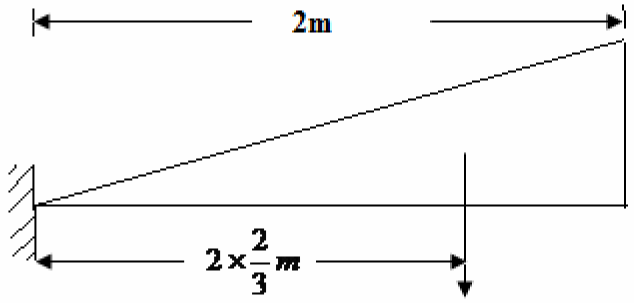
IES-13. Ans. (b)

Cantilever Carrying load Whose Intensity varies

IES-14. **A cantilever beam of 2m length supports a triangularly distributed load over its entire length, the maximum of which is at the free end. The total load is 37.5 kN.What is the bending moment at the fixed end? [IES 2007]**

(a) 50×10^6 N mm (b) 12.5×10^6 N mm (c) 100×10^6 N mm (d) 25×10^6 N mm

IES-14. Ans. (a)



$M = 37.5\times\frac{4}{3}\text{ KNm} = 50\times10^6\text{ Nmm}$

Simply Supported Beam Carrying Concentrated Load

IES-15. Assertion (A): If the bending moment along the length of a beam is constant, then the beam cross section will not experience any shear stress. [IES-1998]
Reason (R): The shear force acting on the beam will be zero everywhere along the length.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-15. Ans. (a)

IES-16. Assertion (A): If the bending moment diagram is a rectangle, it indicates that the beam is loaded by a uniformly distributed moment all along the length.
Reason (R): The BMD is a representation of internal forces in the beam and not the moment applied on the beam. [IES-2002]

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-16. Ans. (d)

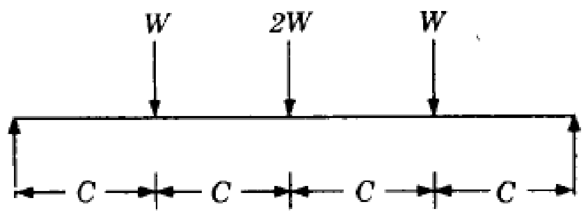
IES-17. The maximum bending moment in a simply supported beam of length L loaded by a concentrated load W at the midpoint is given by [IES-1996]

(a) WL (b) $\frac{WL}{2}$ (c) $\frac{WL}{4}$ (d) $\frac{WL}{8}$

IES-17. Ans. (c)

IES-18. A simply supported beam is loaded as shown in the above figure. The maximum shear force in the beam will be

(a) Zero (b) W
(c) 2W (d) 4W



[IES-1998]

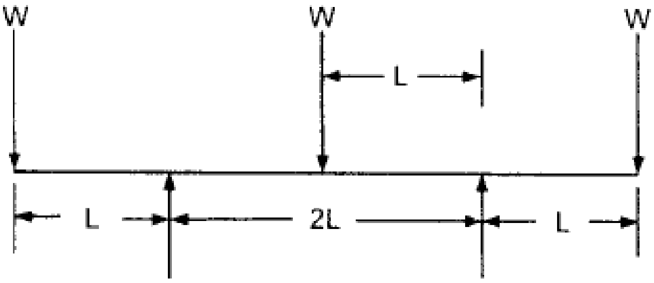
IES-18. Ans. (c)

IES-19. If a beam is subjected to a constant bending moment along its length, then the shear force will [IES-1997]

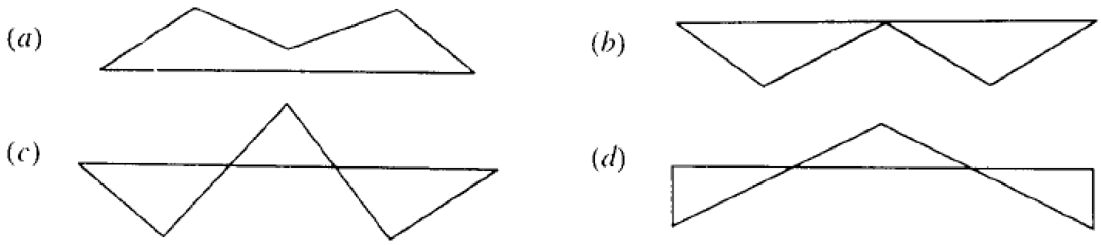
(a) Also have a constant value everywhere along its length
(b) Be zero at all sections along the beam
(c) Be maximum at the centre and zero at the ends (d) zero at the centre and maximum at the ends

IES-19. Ans. (b)

IES-20. **A loaded beam is shown in the figure. The bending moment diagram of the beam is best represented as:**



[IES-2000]

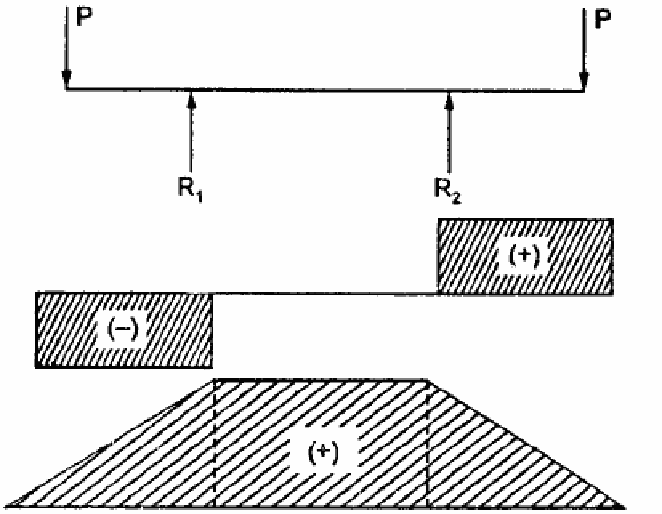


IES-20. Ans. (a)

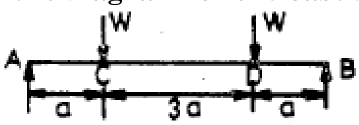
IES-21. **A simply supported beam has equal over-hanging lengths and carries equal concentrated loads P at ends. Bending moment over the length between the supports** **[IES-2003]**

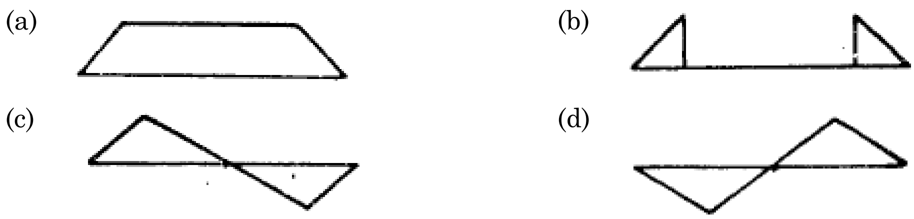
(a) Is zero
(b) Is a non-zero constant
(c) Varies uniformly from one support to the other
(d) Is maximum at mid-span

IES-21. Ans. (b)



IES-22. **The bending moment diagram for the case shown below will be q as shown in**



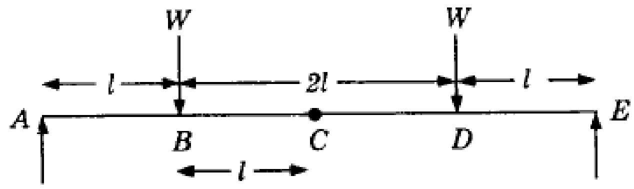


[IES-1992]

IES-22. Ans. (a)

IES-23. Which one of the following portions of the loaded beam shown in the given figure is subjected to pure bending?

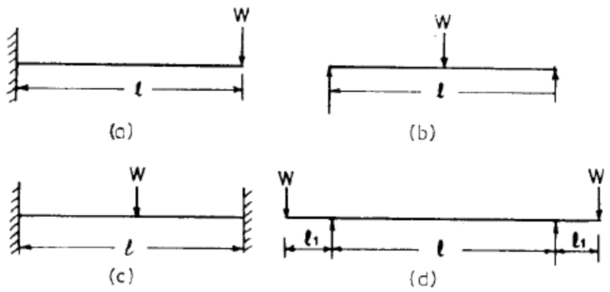
(a) AB (b)DE
(c) AE (d) BD



[IES-1999]

IES-23. Ans. (d) Pure bending takes place in the section between two weights W

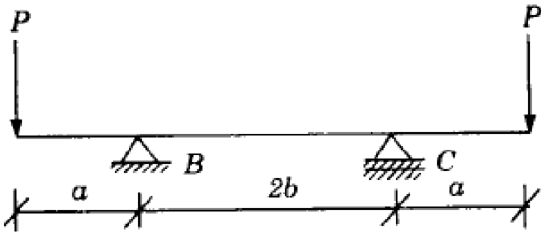
IES-24. Constant bending moment over span "l" will occur in [IES-1995]



IES-24. Ans. (d)

IES-25. For the beam shown in the above figure, the elastic curve between the supports B and C will be:

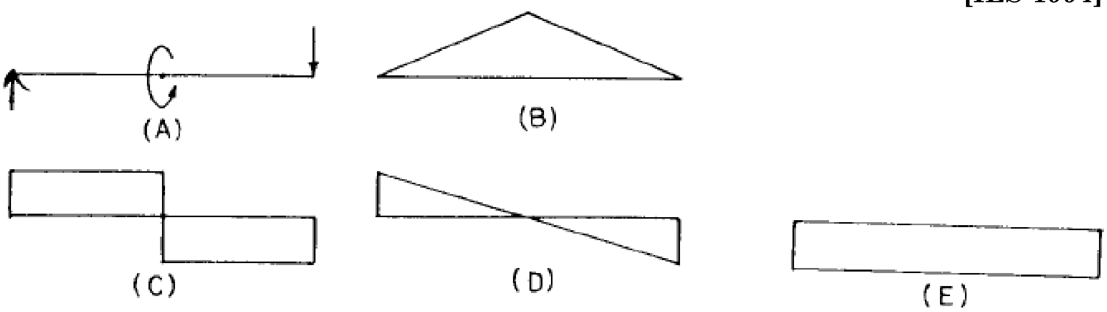
(a) Circular (b) Parabolic
(c) Elliptic (d) A straight line



[IES-1998]

IES-25. Ans. (b)

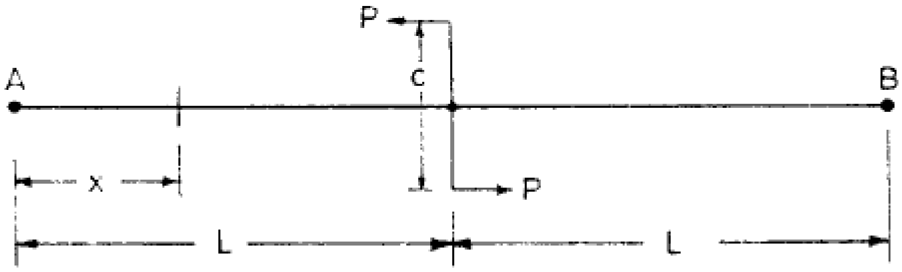
IES-26. A beam is simply supported at its ends and is loaded by a couple at its mid-span as shown in figure A. Shear force diagram for the beam is given by the figure. [IES-1994]



(a) B (b) C (c) D (d) E

IES-26. Ans. (d)

IES-27. A beam AB is hinged-supported at its ends and is loaded by couple P.c. as shown in the given figure. The magnitude or shearing force at a section x of the beam is: [IES-1993]



(a) 0 (b) P (c) P/2L (d) P.c./2L

IES-27. Ans. (d) If F be the shearing force at section x (at point A), then taking moments about B, F x 2L = Pc

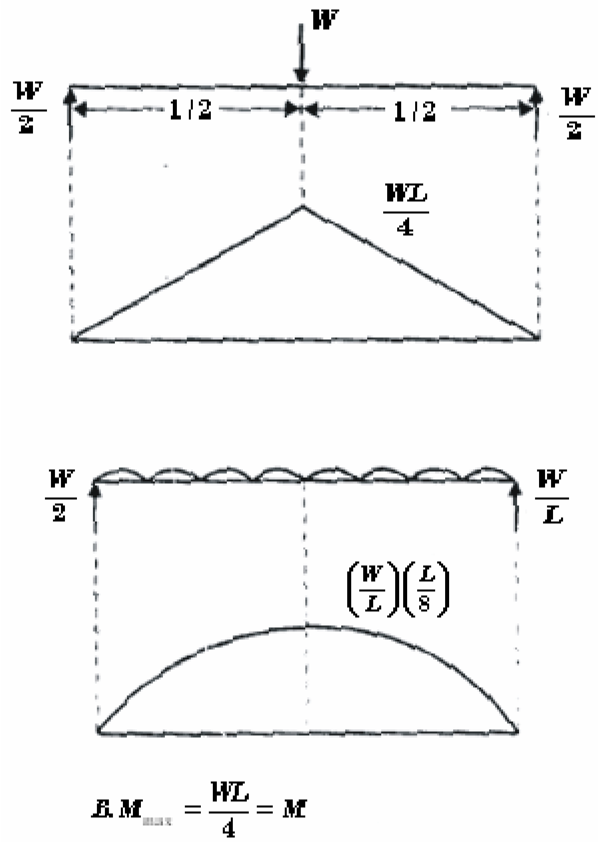
$$or \quad F = \frac{Pc}{2L} \qquad \text{Thus shearing force in zone x} = \frac{Pc}{2L}$$

Simply Supported Beam Carrying a Uniformly Distributed Load

IES-28. A freely supported beam at its ends carries a central concentrated load, and maximum bending moment is M. If the same load be uniformly distributed over the beam length, then what is the maximum bending moment? **[IES-2009]**

(a) M (b) $\frac{M}{2}$ (c) $\frac{M}{3}$ (d) 2M

IES-28. Ans. (b)



$$B.M_{Max} = \frac{WL}{4} = M$$

Where the Load is U.D.L.

Maximum Bending Moment

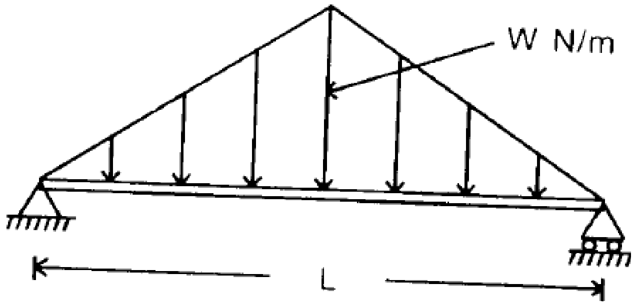
$$= \left(\frac{W}{L}\right)\left(\frac{L^2}{8}\right)$$

$$= \frac{WL}{8} = \frac{1}{2}\left(\frac{WL}{4}\right) = \frac{M}{2}$$

Simply Supported Beam Carrying a Load whose Intensity varies Uniformly from Zero at each End to w per Unit Run at the MiD Span

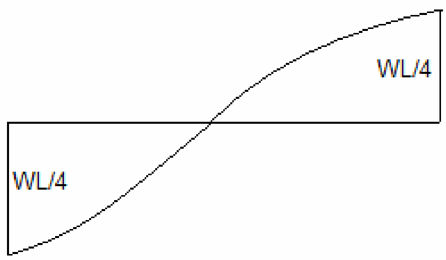
IES-29. **A simply supported beam is subjected to a distributed loading as shown in the diagram given below: What is the maximum shear force in the beam?**

(a) WL/3 (b) WL/2
(c) WL/3 (d) WL/6



[IES-2004]

IES-29. Ans. (d)



$$\text{Total load} = \frac{1}{2} \times L \times W = \frac{WL}{2}$$

$$S_x = \frac{WL}{4} - \frac{1}{2} x . \left(\frac{W}{\frac{L}{2}} \times X \right) = \frac{WL}{4} - \frac{Wx^2}{L}$$

$$S_{max \text{ at } x=0} = \frac{WL}{4}$$

Simply Supported Beam carrying a Load whose Intensity varies

IES-30. **A beam having uniform cross-section carries a uniformly distributed load of intensity q per unit length over its entire span, and its mid-span deflection is δ.**

The value of mid-span deflection of the same beam when the same load is distributed with intensity varying from 2q unit length at one end to zero at the other end is: **[IES-1995]**

(a) 1/3 δ (b) 1/2 δ (c) 2/3 δ (d) δ

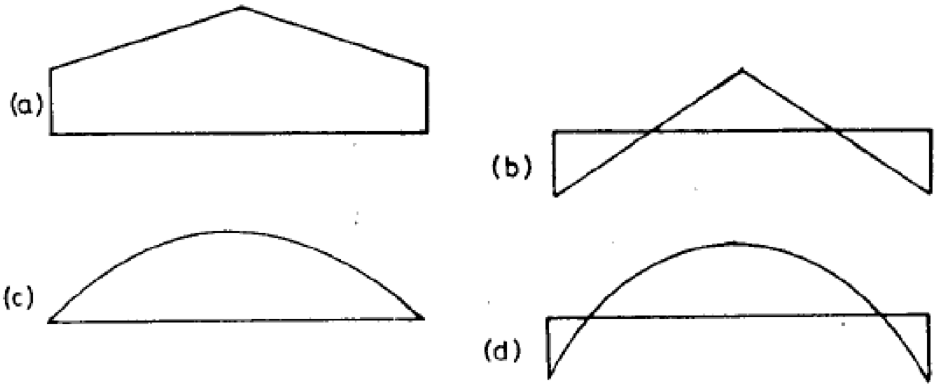
IES-30. Ans. (d)

Simply Supported Beam with Equal Overhangs and carrying a Uniformly Distributed Load

IES-31. **A beam, built-in at both ends, carries a uniformly distributed load over its entire span as shown in figure-I. Which one of the diagrams given below, represents bending moment distribution along the length of the beam?**

[IES-1996]

Fig·I



IES-31. Ans. (d)

The Points of Contraflexure

IES-32. The point· of contraflexure is a point where: [IES-2005]
(a) Shear force changes sign (b) Bending moment changes sign
(c) Shear force is maximum (d) Bending moment is maximum

IES-32. Ans. (b)

IES-33. Match List I with List II and select the correct answer using the codes given below the Lists: [IES-2000]

List-I		**List-II**	
A.	Bending moment is constant	**1.**	Point of contraflexure
B.	Bending moment is maximum or minimum	**2.**	Shear force changes sign
C.	Bending moment is zero	**3.**	Slope of shear force diagram is zero over the portion of the beam
D.	Loading is constant	**4.**	Shear force is zero over the portion of the beam

Code:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	4	1	2	3	(b)	3	2	1	4
(c)	4	2	1	3	(d)	3	1	2	4

IES-33. Ans. (b)

Loading and B.M. diagram from S.F. Diagram

IES-34. The bending moment diagram shown in Fig. I correspond to the shear force diagram in [IES-1999]

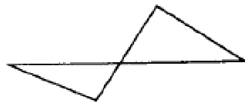
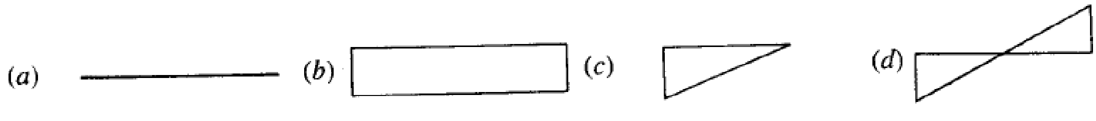
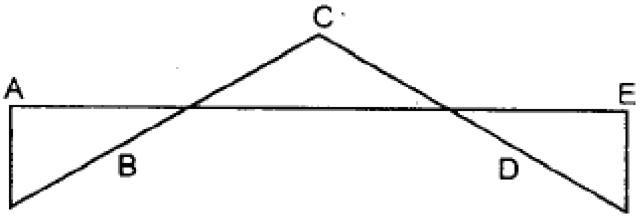


Figure-I

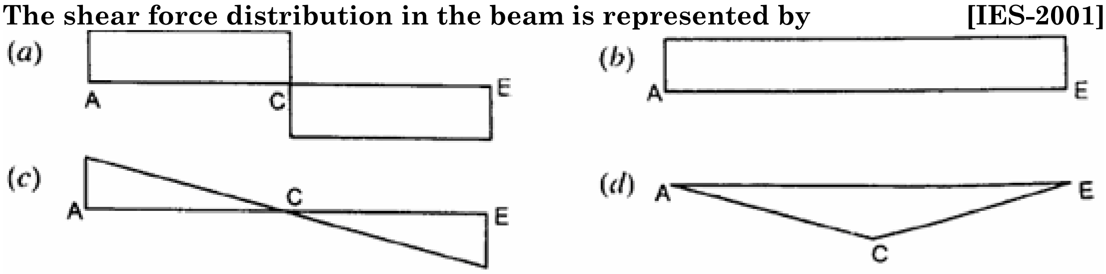


IES-34. Ans. (b) If shear force is zero, B.M. will also be zero. If shear force varies linearly with length, B.M. diagram will be curved line.

IES-35. Bending moment distribution in a built be am is shown in the given



The shear force distribution in the beam is represented by **[IES-2001]**



IES-35. Ans. (a)

IES-36. The given figure shows the shear force diagram for the beam ABCD.

Bending moment in the portion BC of the beam

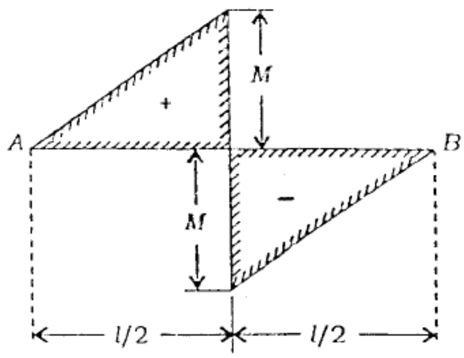
[IES-1996]

(a) Is a non-zero constant
(b) Is zero
(c) Varies linearly from B to C
(d) Varies parabolically from B to C

IES-36. Ans. (a)

IES-37. Figure shown above represents the BM diagram for a simply supported beam. The beam is subjected to which one of the following?

(a) A concentrated load at its mid-length
(b) A uniformly distributed load over its length
(c) A couple at its mid-length
(d) Couple at 1/4 of the span from each end

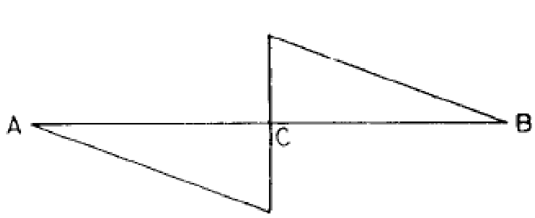


[IES-2006]

IES-37. Ans. (c)

IES-38. If the bending moment diagram for a simply supported beam is of the form given below.
Then the load acting on the beam is:

(a) A concentrated force at C
(b) A uniformly distributed load over the whole length of the beam
(c) Equal and opposite moments applied at A and B
(d) A moment applied at C

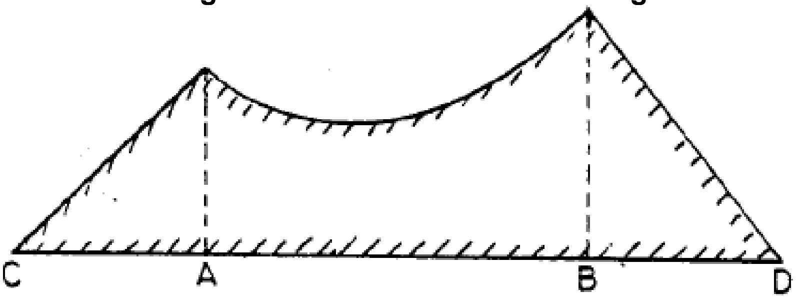


B.M. Diagram

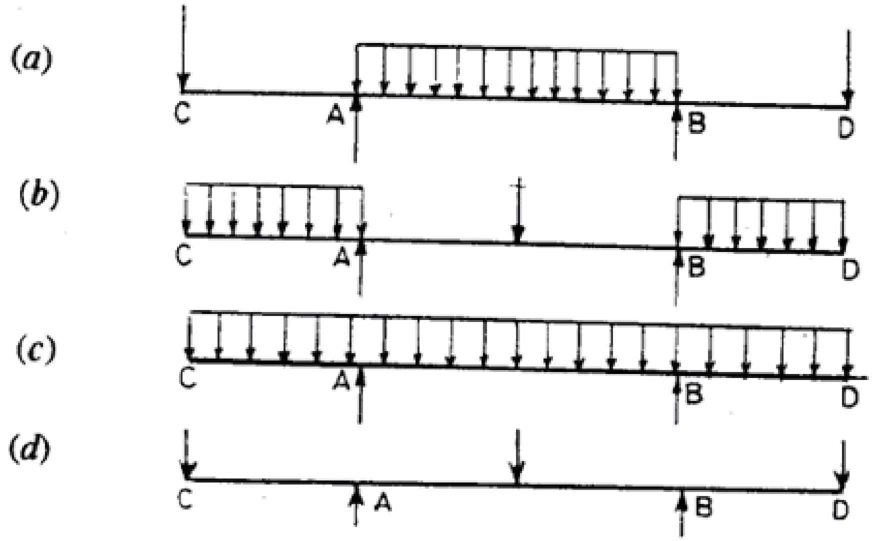
[IES-1994]

IES-38. Ans. (d) A vertical line in centre of B.M. diagram is possible when a moment is applied there.

IES-39. The figure given below shows a bending moment diagram for the beam CABD:



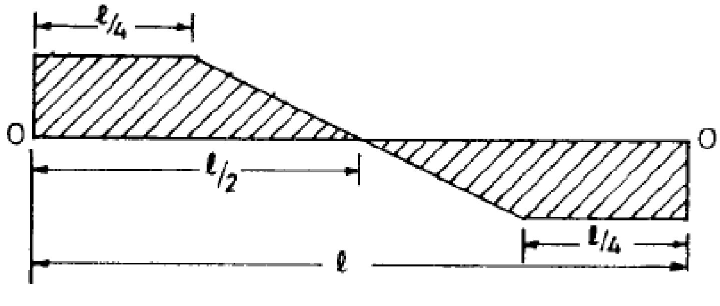
Load diagram for the above beam will be: **[IES-1993]**



IES-39. Ans. (a) Load diagram at (a) is correct because B.M. diagram between A and B is parabola which is possible with uniformly distributed load in this region.

IES-40. The shear force diagram shown in the following figure is that of a **[IES-1994]**

(a) Freely supported beam with symmetrical point load about mid-span.
(b) Freely supported beam with symmetrical uniformly distributed load about mid-span
(c) Simply supported beam with positive and negative point loads symmetrical about the mid-span
(d) Simply supported beam with symmetrical varying load about mid-span



IES-40. Ans. (b) The shear force diagram is possible on simply supported beam with symmetrical varying load about mid span.

Previous 20-Years IAS Questions

Shear Force (S.F.) and Bending Moment (B.M.)

IAS-1. Assertion (A): A beam subjected only to end moments will be free from shearing force. **[IAS-2004]**

Reason (R): The bending moment variation along the beam length is zero.

(a) Both A and R are individually true and R is the correct explanation of A

(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-1. Ans. (a)

IAS-2. Assertion (A): The change in bending moment between two cross-sections of a beam is equal to the area of the shearing force diagram between the two sections. [IAS-1998]
Reason (R): The change in the shearing force between two cross-sections of beam due to distributed loading is equal to the area of the load intensity diagram between the two sections.
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

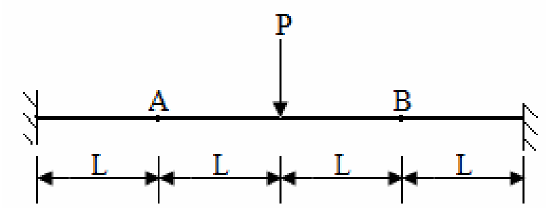
IAS-2. Ans. (b)

IAS-3. The ratio of the area under the bending moment diagram to the flexural rigidity between any two points along a beam gives the change in [IAS-1998]
(a) Deflection (b) Slope (c) Shear force (d) Bending moment

IAS-3. Ans. (b)

Cantilever

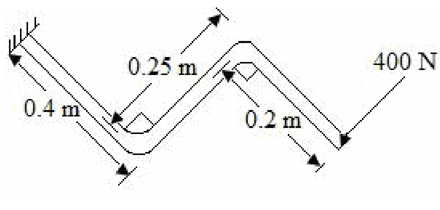
IAS-4. A beam AB of length 2 L having a concentrated load P at its mid-span is hinge supported at its two ends A and B on two identical cantilevers as shown in the given figure. The correct value of bending moment at A is



(a) Zero (b) PLl2
(c) PL (d) 2 PL **[IAS-1995]**

IAS-4. Ans. (a) Because of hinge support between beam *AB* and cantilevers, the bending moment can't be transmitted to cantilever. Thus bending moment at points *A* and *B* is zero.

IAS-5. A load perpendicular to the plane of the handle is applied at the free end as shown in the given figure. The values of Shear Forces (S.F.), Bending Moment (B.M.) and torque at the fixed end of the handle have been determined respectively as 400 N, 340 Nm and 100 by a student. Among these values, those of [IAS-1999]
(a) S.F., B.M. and torque are correct
(b) S.F. and B.M. are correct
(c) B.M. and torque are correct
(d) S.F. and torque are correct



IAS-5. Ans. (d)

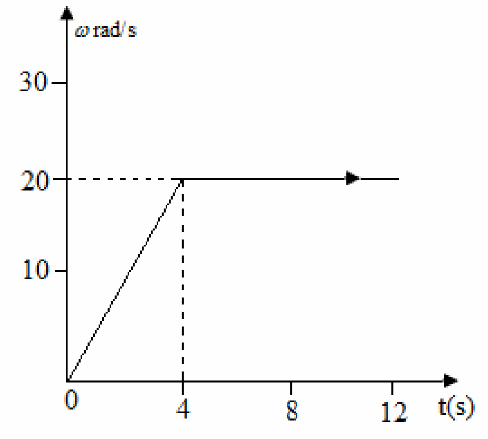
$$\text{S.F} = 400\text{N} \quad \text{and} \quad \text{BM} = 400 \times (0.4 + 0.2) = 240\text{Nm}$$

$$\text{Torque} = 400 \times 0.25 = 100\text{Nm}$$

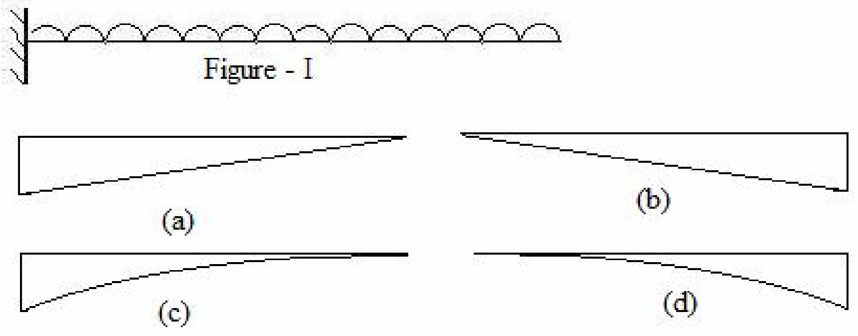
Cantilever with Uniformly Distributed Load

IAS-6. If the SF diagram for a beam is a triangle with length of the beam as its base, the beam is: [IAS-2007]
(a) A cantilever with a concentrated load at its free end
(b) A cantilever with udl over its whole span
(c) Simply supported with a concentrated load at its mid-point
(d) Simply supported with a udl over its whole span

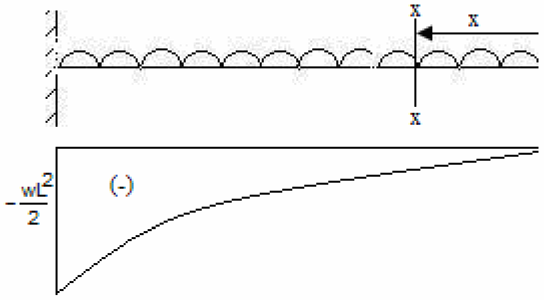
IAS-6. Ans. (b)



IAS-7. **A cantilever carrying a uniformly distributed load is shown in Fig. I. Select the correct R.M. diagram of the cantilever.** **[IAS-1999]**

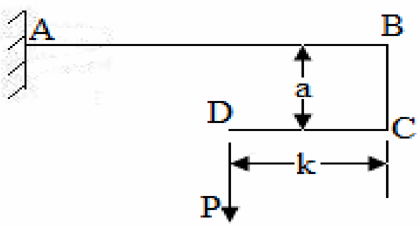


IAS-7. Ans. (c) $M_x = -wx \times \frac{x}{2} = -\frac{wx^2}{2}$



IAS-8. **A structural member ABCD is loaded as shown in the given figure. The shearing force at any section on the length BC of the member is:**

(a) Zero (b) P
(c) Pa/k (d) Pk/a

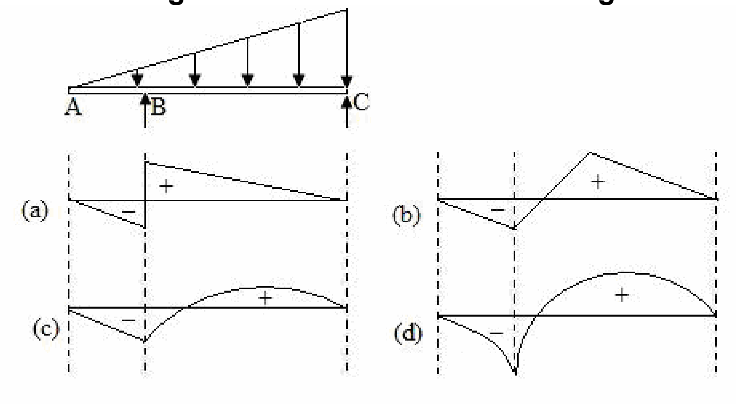


[IAS-1996]

IAS-8. Ans. (a)

Cantilever Carrying load Whose Intensity varies

IAS-9. **The beam is loaded as shown in Fig. I. Select the correct B.M. diagram** **[IAS-1999]**



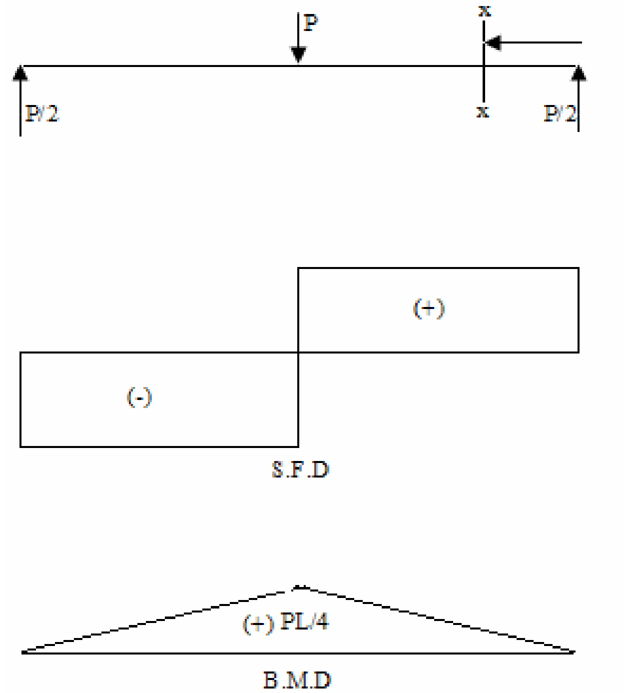
IAS-9. Ans. (d)

Simply Supported Beam Carrying Concentrated Load

IAS-10. Assertion (A): In a simply supported beam carrying a concentrated load at mid-span, both the shear force and bending moment diagrams are triangular in nature without any change in sign. [IAS-1999]
Reason (R): When the shear force at any section of a beam is either zero or changes sign, the bending moment at that section is maximum.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

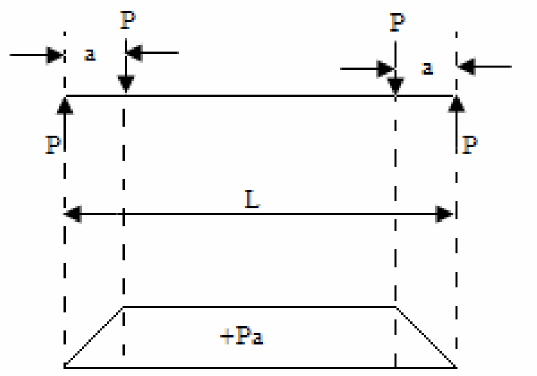
IAS-10. Ans. (d) A is false.



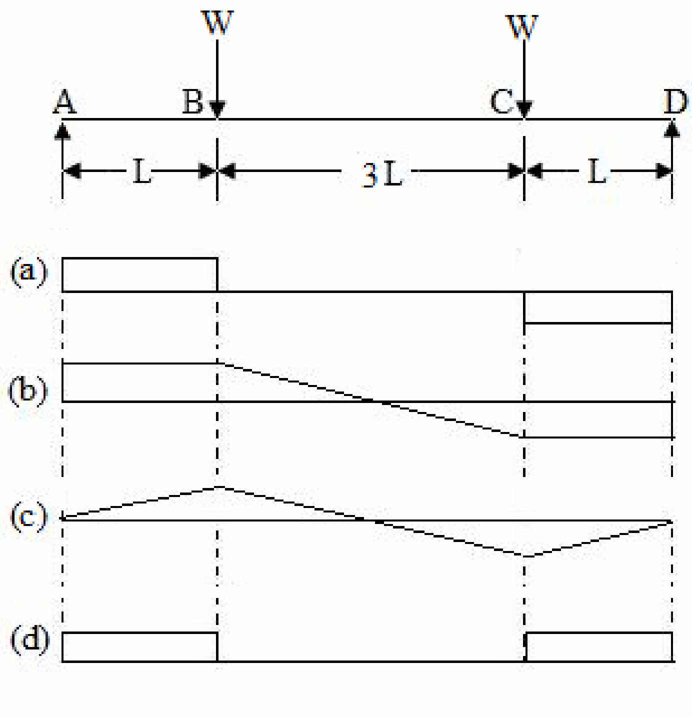
IAS-11. For the shear force to be uniform throughout the span of a simply supported beam, it should carry which one of the following loadings? [IAS-2007]

(a) A concentrated load at mid-span
(b) Udl over the entire span
(c) A couple anywhere within its span
(d) Two concentrated loads equal in magnitude and placed at equal distance from each support

IAS-11. Ans. (d) It is a case of pure bending.



IAS-12. **Which one of the following figures represents the correct shear force diagram for the loaded beam shown in the given figure I?** **[IAS-1998; IAS-1995]**



IAS-12. Ans. (a)

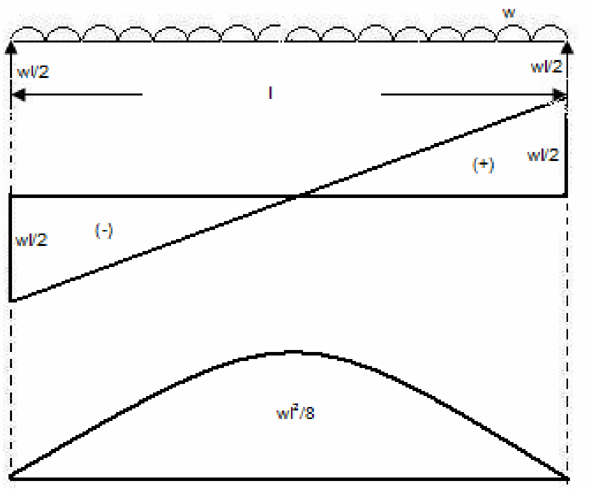
Simply Supported Beam Carrying a Uniformly Distributed Load

IAS-13. **For a simply supported beam of length fl' subjected to downward load of uniform intensity w, match List-I with List-II and select the correct answer using the codes given below the Lists:** **[IAS-1997]**

List-I		**List-II**	
A.	Slope of shear force diagram	**1.**	$\dfrac{5wl^4}{384EI}$
B.	Maximum shear force	**2.**	w
C.	Maximum deflection	**3.**	$\dfrac{wl^4}{8}$
D.	Magnitude of maximum bending moment	**4.**	$\dfrac{wl}{2}$

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	1	2	3	4	(b)	3	1	2	4
(c)	3	2	1	4	(d)	2	4	1	3

IAS-13. Ans. (d)



Simply Supported Beam Carrying a Load whose Intensity varies Uniformly from Zero at each End to w per Unit Run at the MiD Span

IAS-14. A simply supported beam of length 'l' is subjected to a symmetrical uniformly varying load with zero intensity at the ends and intensity w (load per unit length) at the mid span. What is the maximum bending moment? [IAS-2004]

(a) $\frac{3wl^2}{8}$ (b) $\frac{wl^2}{12}$ (c) $\frac{wl^2}{24}$ (d) $\frac{5wl^2}{12}$

IAS-14. Ans. (b)

Simply Supported Beam carrying a Load whose Intensity varies

IAS-15. A simply supported beam of span l is subjected to a uniformly varying load having zero intensity at the left support and w N/m at the right support. The reaction at the right support is: [IAS-2003]

(a) $\frac{wl}{2}$ (b) $\frac{wl}{5}$ (c) $\frac{wl}{4}$ (d) $\frac{wl}{3}$

IAS-15. Ans. (d)

Simply Supported Beam with Equal Overhangs and carrying a Uniformly Distributed Load

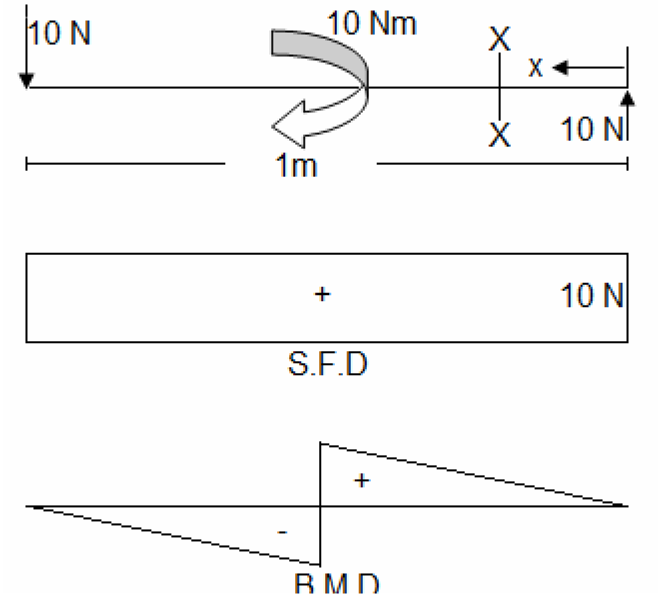
IAS-16. Consider the following statements for a simply supported beam subjected to a couple at its mid-span: [IAS-2004]

1. Bending moment is zero at the ends and maximum at the centre
2. Bending moment is constant over the entire length of the beam
3. Shear force is constant over the entire length of the beam
4. Shear force is zero over the entire length of the beam

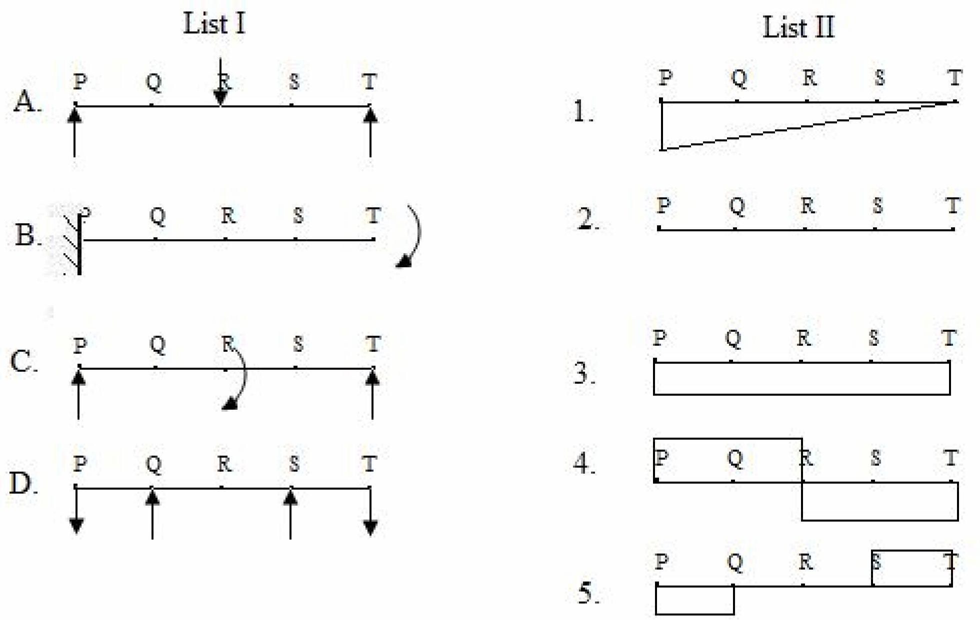
Which of the statements given above are correct?

(a) 1, 3 and 4 (b) 2, 3 and 4 (c) 1 and 3 (d) 2 and 4

IAS-16. Ans. (c)



IAS-17. **Match List-I (Beams) with List-II (Shear force diagrams) and select the correct answer using the codes given below the Lists:** **[IAS-2001]**



Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	4	2	5	3	(b)	1	4	5	3
(c)	1	4	3	5	(d)	4	2	3	5

IAS-17. Ans. (d)

The Points of Contraflexure

IAS-18. **A point, along the length of a beam subjected to loads, where bending moment changes its sign, is known as the point of** **[IAS-1996]**

(a) Inflexion (b) Maximum stress (c) Zero shear force (d) Contra flexure

IAS-18. Ans. (d)

IAS-19. **Assertion (A): In a loaded beam, if the shear force diagram is a straight line parallel to the beam axis, then the bending moment is a straight line inclined to the beam axis.** **[IAS 1994]**

Reason (R): When shear force at any section of a beam is zero or changes sign, the bending moment at that section is maximum.

(a) Both A and R are individually true and R is the correct explanation of A

(b) Both A and R are individually true but R is **NOT** the correct explanation of A

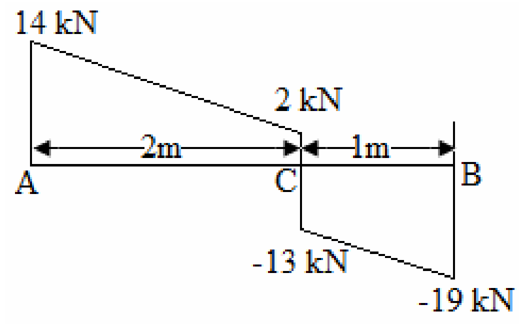
(c) A is true but R is false
(d) A is false but R is true

IAS-19. Ans. (b)

Loading and B.M. diagram from S.F. Diagram

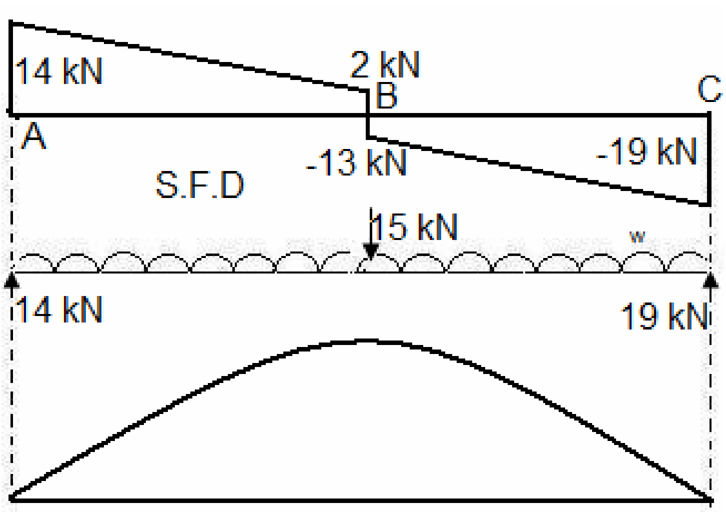
IAS-20. The shear force diagram of a loaded beam is shown in the following figure:
The maximum Bending Moment of the beam is:

(a) 16 kN-m (b) 11 kN-m
(c) 28 kN-m (d) 8 kN-m

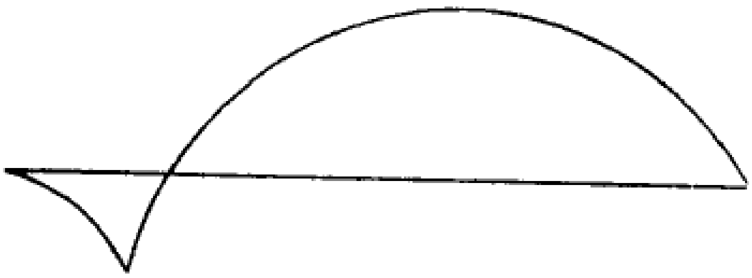


[IAS-1997]

IAS-20. Ans. (a)

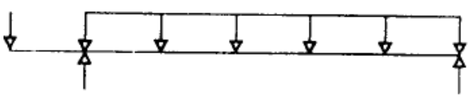


IAS-21. The bending moment for a loaded beam is shown below: [IAS-2003]

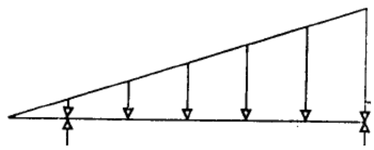


The loading on the beam is represented by which one of the followings diagrams?

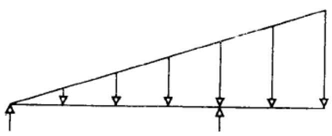
(a)



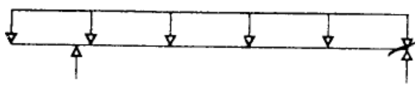
(b)



(c)

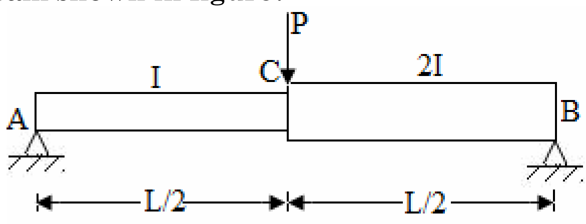


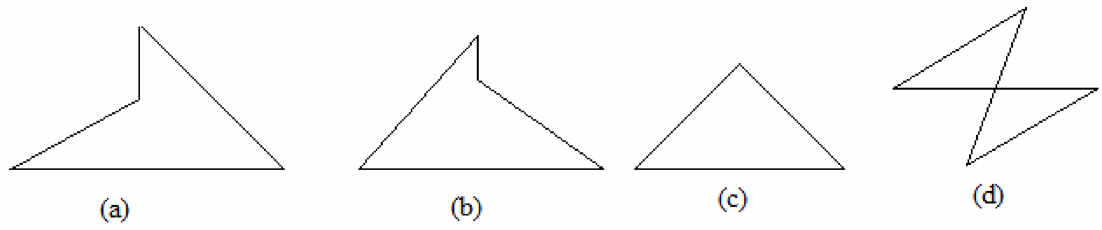
(d)



IAS-21. Ans. (d)

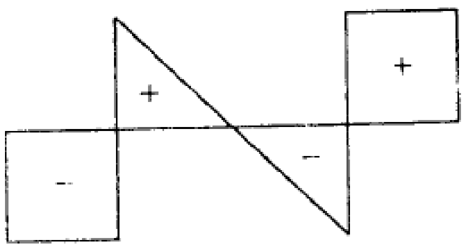
IAS-22. Which one of the given bending moment diagrams correctly represents that of the loaded beam shown in figure? [IAS-1997]





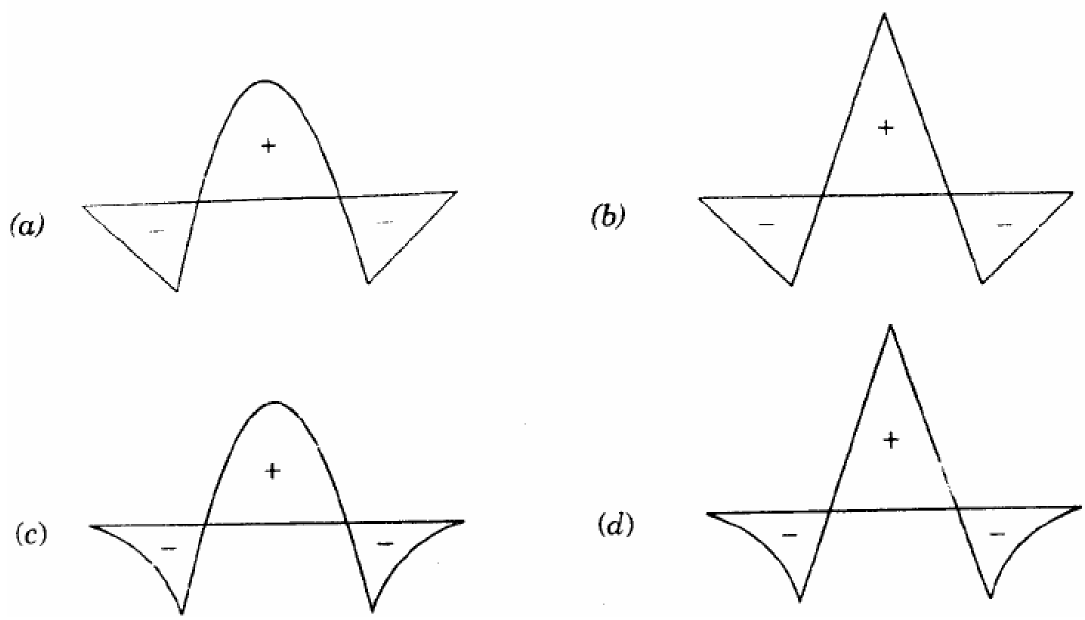
IAS-22. Ans. (c) Bending moment does not depends on moment of inertia.

IAS-23.



The shear force diagram is shown above for a loaded beam. The corresponding bending moment diagram is represented by

[IAS-2003]



IAS-23. Ans. (a)

IAS-24. The bending moment diagram for a simply supported beam is a rectangle over a larger portion of the span except near the supports. What type of load does the beam carry? [IAS-2007]

(a) A uniformly distributed symmetrical load over a larger portion of the span except near the supports
(b) A concentrated load at mid-span
(c) Two identical concentrated loads equidistant from the supports and close to mid-point of the beam

(d) Two identical concentrated loads equidistant from the mid-span and close to supports

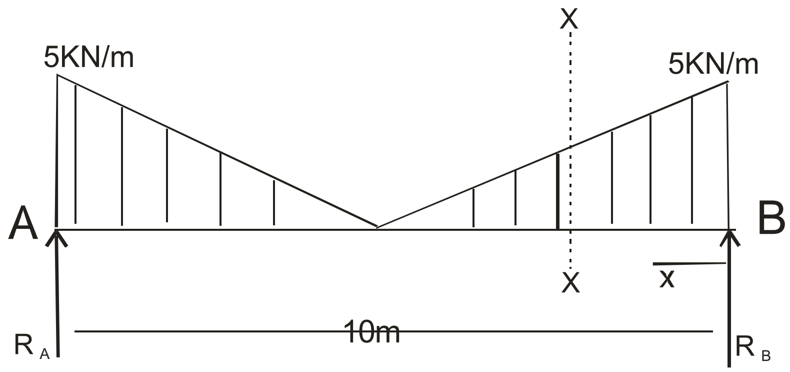
IAS-24. Ans. (d)

Previous Conventional Questions with Answers

Conventional Question IES-2005

Question: **A simply supported beam of length 10 m carries a uniformly varying load whose intensity varies from a maximum value of 5 kN/m at both ends to zero at the centre of the beam. It is desired to replace the beam with another simply supported beam which will be subjected to the same maximum 'bending moment' and 'shear force' as in the case of the previous one. Determine the length and rate of loading for the second beam if it is subjected to a uniformly distributed load over its whole length. Draw the variation of 'SF' and 'BM' in both the cases.**

Answer:



Total load on beam $=5\times\frac{10}{2}=25\,kN$

$\therefore R_A=R_B=\frac{25}{2}=12.5\,kN$

Take a section X-X from B at a distance x.

For $0\le x\le 5m$ we get rate of loading

$\omega=a+bx$ [*as* lineary varying]

at x=0, ω=5 kN/m

and at x = 5, $\omega=0$

These two bounday condition gives a = 5 and b = -1

$\therefore \omega=5-x$

We know that shear force(V), $\frac{dV}{dx}=-\omega$

or $V=\int -\omega dx=-\int(5-x)dx=-5x+\frac{x^2}{2}+c_1$

at x = 0, F =12.5 kN (R_B) so $c_1=12.5$

$$\therefore V=-5x+\frac{x^2}{2}+12.5$$

It is clear that maximum S.F = 12.5 kN

For a beam $\frac{dM}{dx} = V$

$or,\ M = \int Vdx = \int(-5x + \frac{x^2}{2} + 12.5)dx = -\frac{5x^2}{2} + \frac{x^3}{6} + 12.5x + C_2$

at x = 0, M = 0 gives $C_2 = 0$

$M = 12.5x - 2.5x^2 + x^3/6$

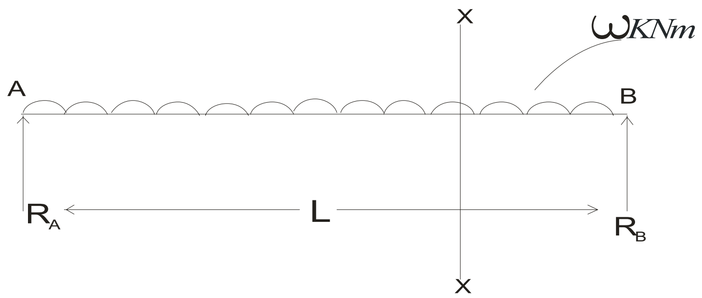
for Maximum bending moment at $\frac{dM}{dx} = 0$

$\text{or} -5x + \frac{x^2}{2} + 12.5 = 0$

$or, x^2 - 10x + 25 = 0$

$or, x = 5$ means at centre.

So, $M_{max} = 12.5 \times 2.5 - 2.5 \times 5^2 + 5^3/6 = 20.83$ kNm



Now we consider a simply supported beam carrying uniform distributed load over whole length (ω KN/m).

Here $R_A = R_B = \frac{WL}{2}$

$S.F.at$ section X-X

$V_x = +\frac{W\ell}{2} - \omega x$

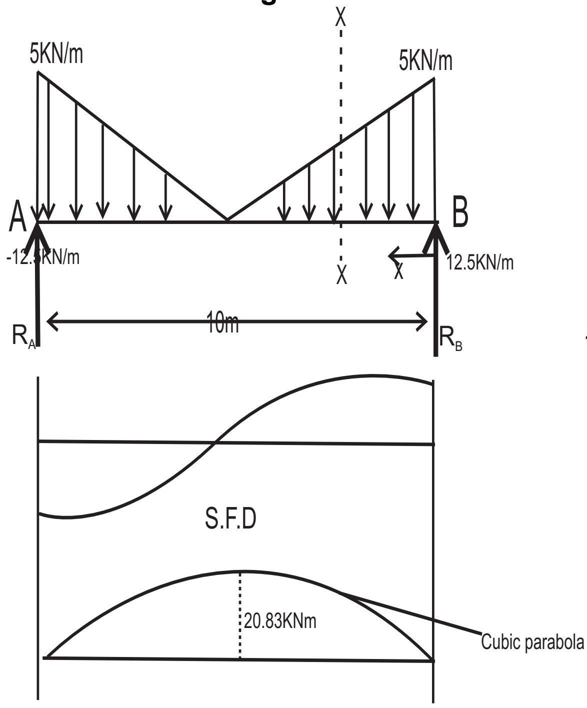
$V_{max} = 12.5\, kN$

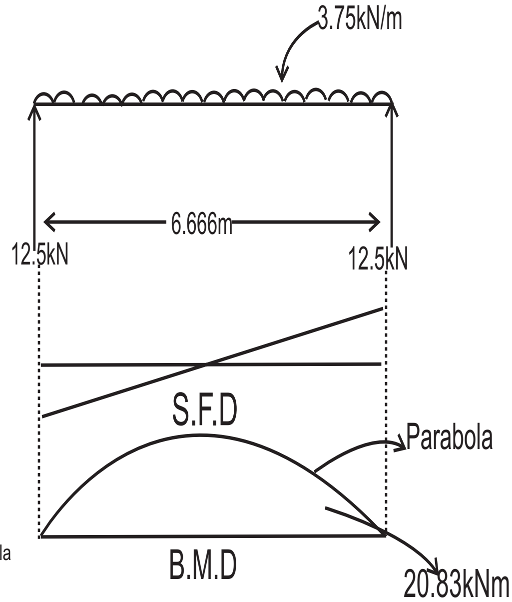
$B.M\, at$ section X-X

$M_x = +\frac{W\ell}{2}x - \frac{Wx^2}{2}$

$\frac{dM_x}{dx} = \frac{WL}{2} - \frac{\omega}{2} \times \left(\frac{L}{2}\right)^2 = \frac{WL^2}{8} = 20.83 - - - -(ii)$

$Solving(i)\,\&\,(ii)$ we get L=6.666m and ω=3.75kN/m





Conventional Question IES-1996

Question: **A Uniform beam of length L is carrying a uniformly distributed load w per unit length and is simply supported at its ends. What would be the maximum bending moment and where does it occur?**

Answer: By symmetry each support reaction is equal i.e. $R_A = R_B = \frac{W\ell}{2}$

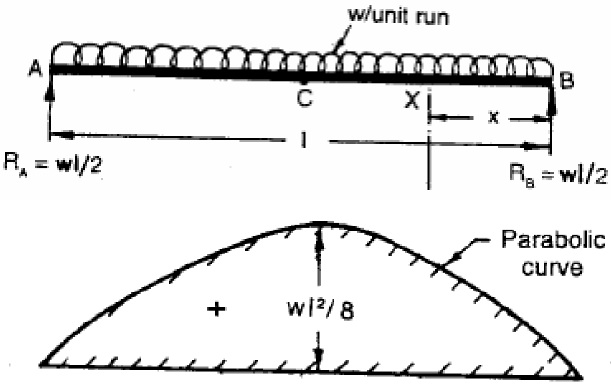
B.M at the section x-x is

$M_x = +\frac{W\ell}{2}x - \frac{Wx^2}{2}$

For the B.M to be maximum we have to $\frac{dM_x}{dx} = 0$ that gives.

$+\frac{W\ell}{2} - \omega x = 0$

or $x = \ell/2$ i.e. at mid point.

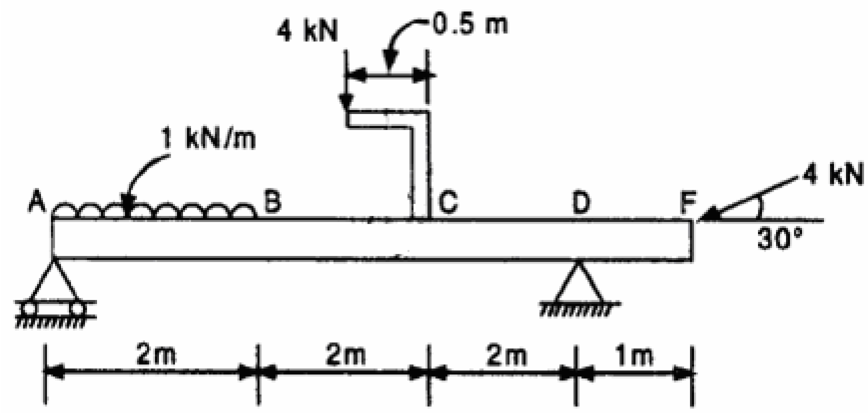


Bending Moment Diagram

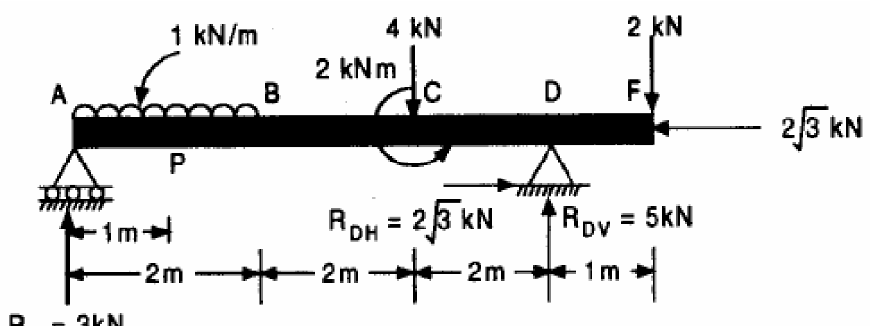
And $M_{max} = \frac{\omega\ell}{2} \times \ell/2 - \frac{\omega}{2} \times \left[\frac{\ell}{2}\right]^2 = +\frac{w\ell^2}{8}$

Conventional Question AMIE-1996

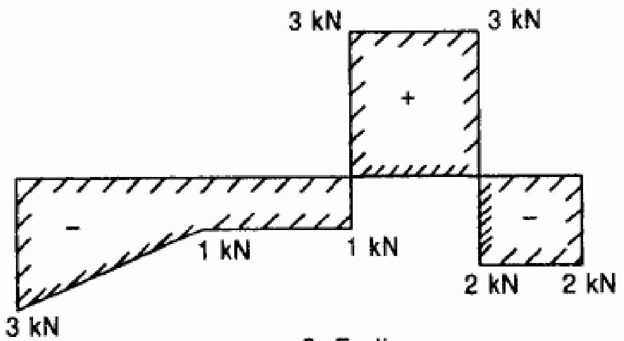
Question: **Calculate the reactions at A and D for the beam shown in figure. Draw the bending moment and shear force diagrams showing all important values.**



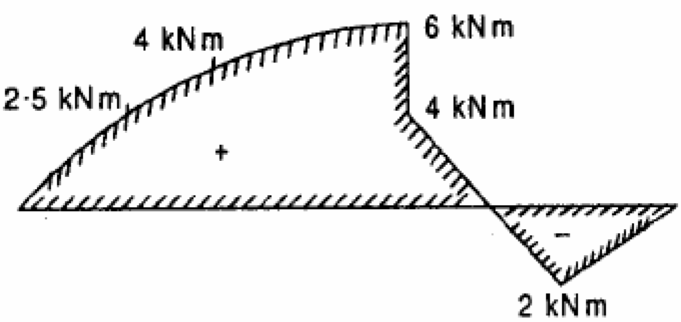
***Answer*:** Equivalent figure below shows an overhanging beam ABCDF supported by a roller support at A and a hinged support at D. In the figure, a load of 4 kN is applied through a bracket 0.5 m away from the point C. Now apply equal and opposite load of 4 kN at C. This will be equivalent to a anticlockwise couple of the value of (4 x 0.5) = 2 kNm acting at C together with a vertical downward load of 4 kN at C. Show U.D.L. (1 kN/m) over the port AB, a point load of 2 kN vertically downward at F, and a horizontal load of $2\sqrt{3}$ kN as shown.



Beam



S. F. diagram



B.M. diagram

For reaction and A and D.

Let ue assume R_A= reaction at roller A.

R_{DV} vertically component of the reaction at the hinged support D, and

R_{DH} horizontal component of the reaction at the hinged support D.

Obviously $R_{DH} = 2\sqrt{3}$ kN $(\rightarrow)$

In order to determine R_A, takings moments about D, we get

$$R_A \times 6 + 2 \times 1 = 1 \times 2 \times \left(\frac{2}{2} + 2 + 2\right) + 2 + 4 \times 2$$

or $\quad R_A = 3\text{kN}$

Also $R_A + R_{DV} = (1 \times 2) + 4 + 2 = 8$

or $\quad R_{DV} = 5\text{kN vetrically upward}$

$\therefore$ Reaction at D, $R_D = \sqrt{(R^2{}_{DV}) + (R_{DH})^2} = \sqrt{5^2 + (2\sqrt{3})^2} = 6.08\text{kN}$

Inclination with horizontal $= \theta = \tan^{-1}\dfrac{5}{2\sqrt{3}} = 55.3^0$

S.F. Calculation:

$$V_F = -2\text{kN}$$
$$V_D = -2 + 5 = 3\text{kN}$$
$$V_C = 3 - 4 = -1\text{kN}$$
$$V_B = -1\text{kN}$$
$$V_A = -1 - (1 \times 2) = -3\text{kN}$$

B.M. Calculation:

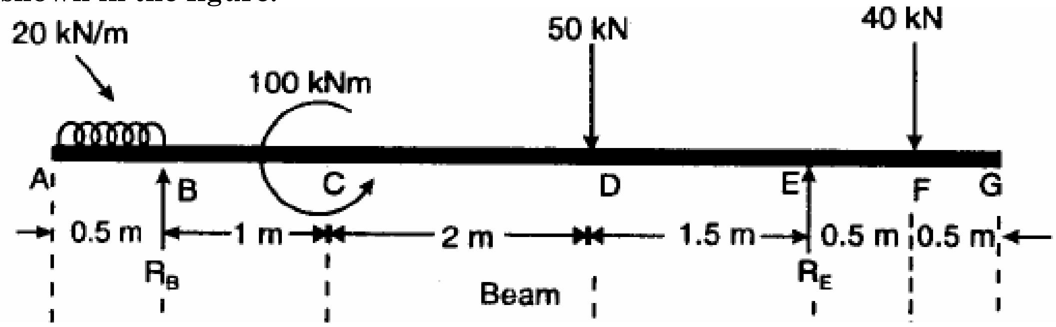
$$M_F = 0$$
$$M_D = -2 \times 1 = -2\text{kNm}$$
$$M_C = \left[-2(1+2) + 5 \times 2\right] + 2 = 6\text{kNm}$$

The bending moment increases from 4 kNm in $(\text{i,e.,} -2(1+2) + 5 \times 2)$ to 6kNm as shown

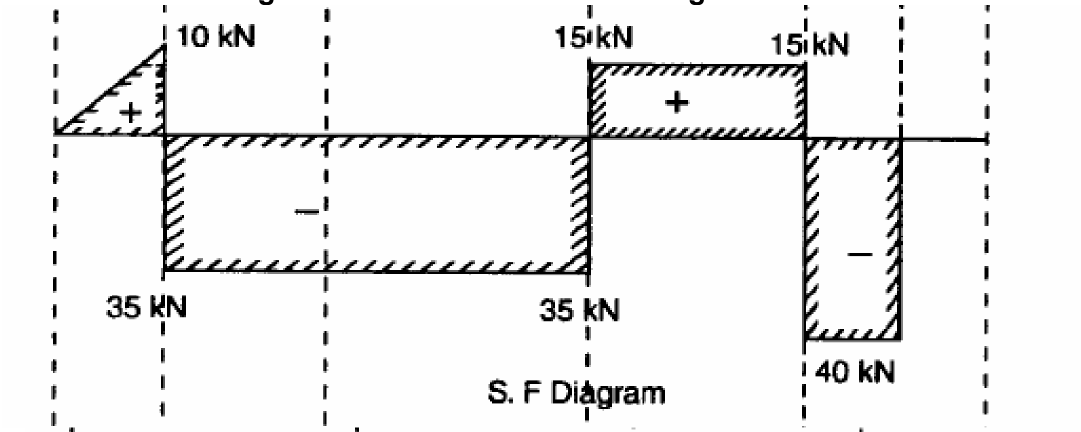
$$M_B = -2(1+2+2) + 5(+2) - 4 \times 2 + 2 = 4\text{kNm}$$
$$M_P = -2\left(1+2+2+\frac{2}{2}\right) + 5(2+2+1) - 4(2+1) + 2 - 1 \times 1 \times \frac{1}{2}$$
$$= 2.5\text{kNm}$$
$$M_A = 0$$

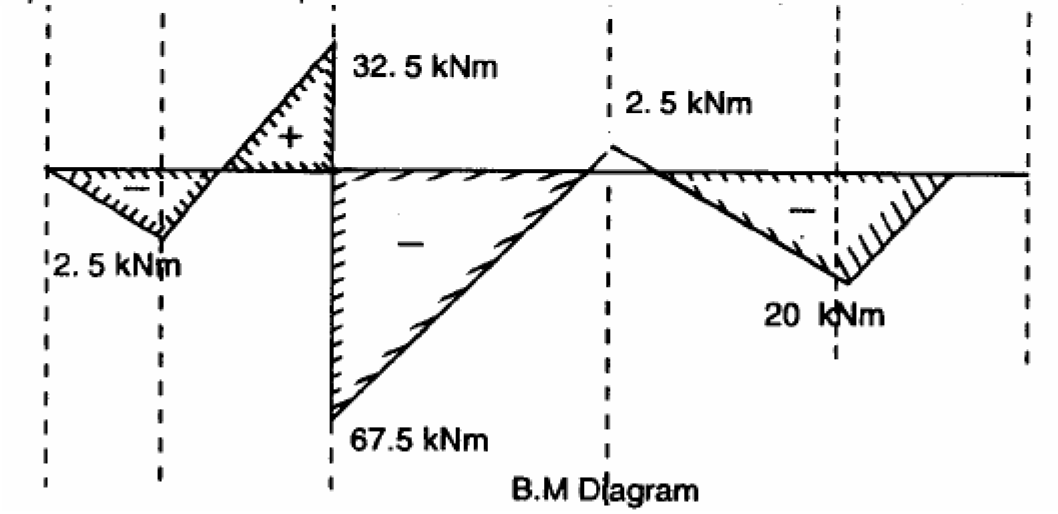
Conventional Question GATE-1997

Question**:** **Construct the bending moment and shearing force diagrams for the beam shown in the figure.**



Answer**:**





Calculation: First find out reaction at B and E.
Taking moments, about B, we get

$$R_E \times 4.5 + 20 \times 0.5 \times \frac{0.5}{2} + 100 = 50 \times 3 + 40 \times 5$$

or $R_E = 55\text{kN}$

Also, $R_B + R_E = 20 \times 0.5 + 50 + 40$

or $R_B = 45\text{kN}$ $[\because R_E = 55\text{kN}]$

S.F. Calculation: $V_F = -40\text{kN}$

$V_E = -40 + 55 = 15\text{kN}$

$V_D = 15 - 50 = -35\text{kN}$

$V_B = -35 + 45 = 10\text{kN}$

B.M. Calculation: $M_G = 0$

$M_F = 0$

$M_E = -40 \times 0.5 = -20\text{kNm}$

$M_D = -40 \times 2 + 55 \times 1.5 = 2.5\text{kNm}$

$M_C = -40 \times 4 + 55 \times 3.5 - 50 \times 2 = -67.5\text{kNm}$

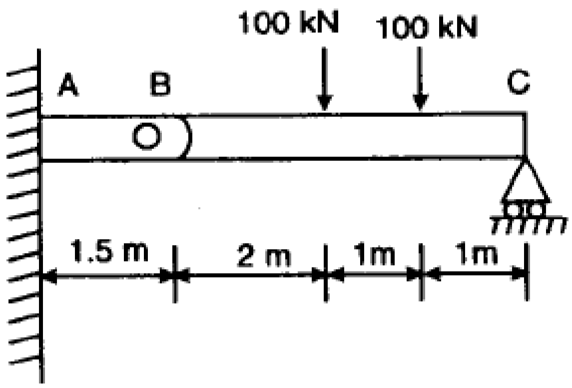
The bending moment increases from -62.5kNm to 100.

$$M_B = -20 \times 0.5 \times \frac{0.5}{2} = -2.5\text{kNm}$$

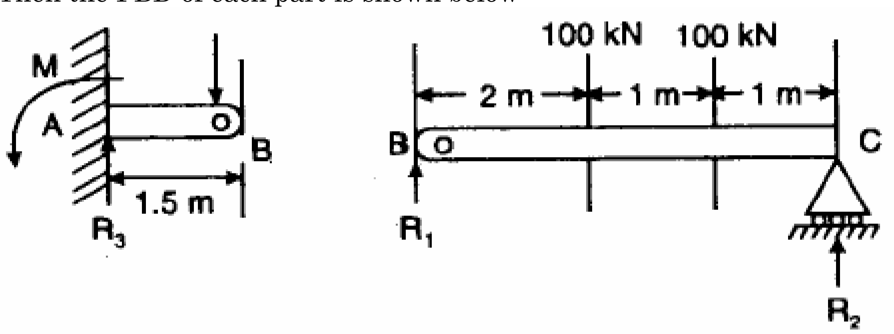
Conventional Question GATE-1996

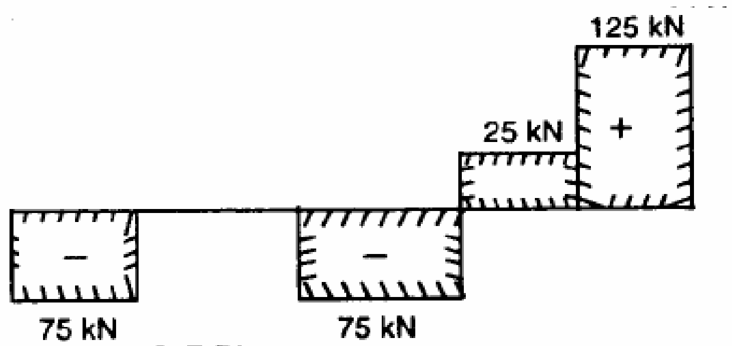
Question: **Two bars AB and BC are connected by a frictionless hinge at B. The assembly is supported and loaded as shown in figure below. Draw the shear force and bending moment diagrams for the combined beam AC. clearly labelling the**

important values. Also indicate your sign convention.

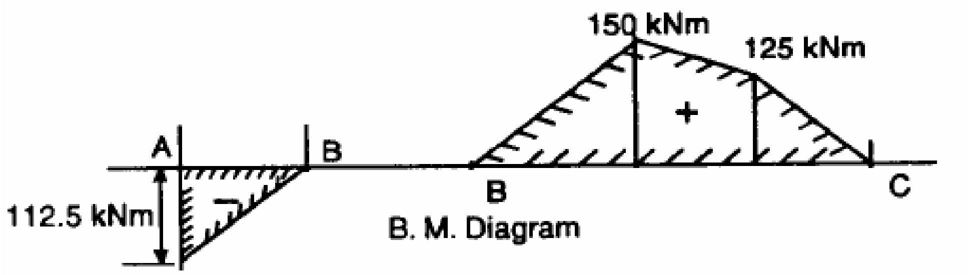


***Answer*:** There shall be a vertical reaction at hinge B and we can split the problem in two parts. Then the FBD of each part is shown below





S. F. Diagram



B. M. Diagram

Calculation: Referring the FBD, we get,

$F_y = 0$, and $R_1 + R_2 = 200\text{kN}$

From $\sum M_B = 0, 100 \times 2 + 100 \times 3 - R_2 \times 4 = 0$

or $R_2 = \dfrac{500}{4} = 125\text{kN}$

$\therefore$ $R_1 = 200 - 125 = 75\text{kN}$

Again, $R_3 = R_1 = 75\text{kN}$

and $M = 75 \times 1.5 = 112.5\text{kNm}.$

Conventional Question IES-1998

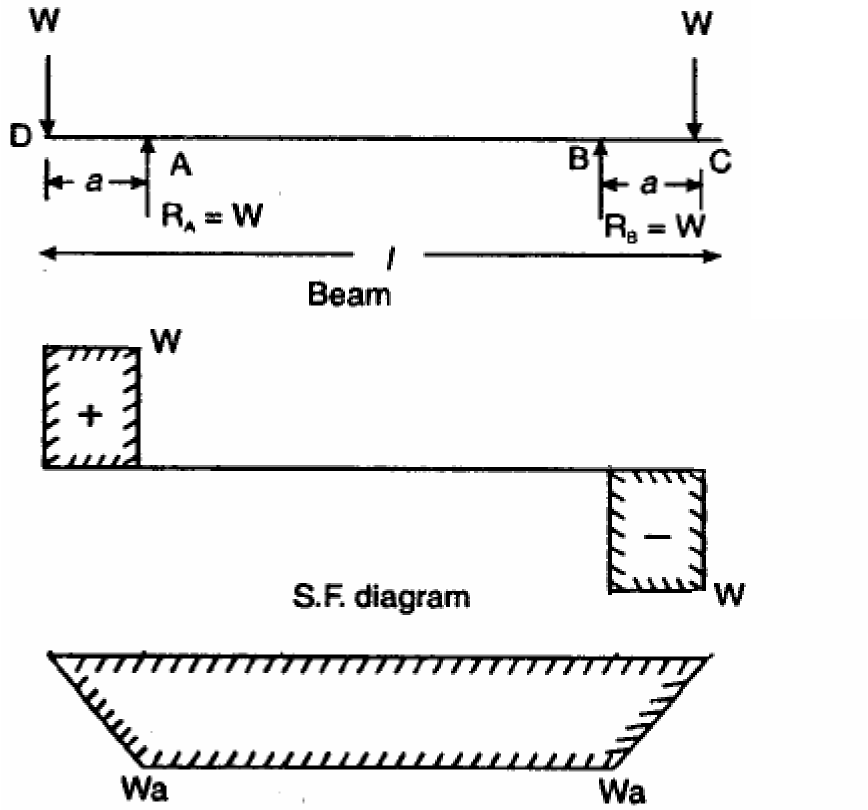
Question: **A tube 40 mm outside diameter; 5 mm thick and 1.5 m long simply supported at 125 mm from each end carries a concentrated load of 1 kN at each extreme end.**

(i) Neglecting the weight of the tube, sketch the shearing force and bending moment diagrams;

(ii) Calculate the radius of curvature and deflection at mid-span. Take the modulus of elasticity of the material as 208 GN/m²

Answer: ***(i)*** Given, $d_0 = 40\,mm = 0.04\,m;\ d_i = d_0 - 2t = 40 - 2 \times 5 = 30\,mm = 0.03\,m;$

$W = 1kN;\ E = 208GN/m^2 = 208 \times 10^2 N/m^2;\ l = 1.5;\ a = 125\,mm = 0.125\,m$



Beam

S.F. diagram

B.M. diagram

Calculation:

(ii) Radius of coordinate R

As per bending equation:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\text{or}\quad R = \frac{EI}{M} \qquad ---(i)$$

$$\text{Here}, M = W \times a = 1 \times 10^3 \times 0.125 = 125Nm$$

$$I = \frac{\pi}{64}\left(d_0^4 - d_i^4\right)$$

$$= \frac{\pi}{64}\left[(0.04)^4 - (0.03)^4\right] = 8.59 \times 10^{-8}\,m^4$$

Substituting the values in equation(i), we get

$$R = \frac{208 \times 10^8 \times 8.59 \times 10^{-8}}{125} = 142.9\,m$$

<u>Deflection at mid − span:</u>

$$EI\frac{d^2y}{dx^2} = M_x = -Wx + W(x - a) = -Wx + Wx - Wa = -Wa$$

Integrating, we get

$$EI\frac{dy}{dx} = -Wax + C_1$$

When, $x = \frac{1}{2}, \frac{dy}{dx} = 0$

$\therefore$ $0 = -Wa\frac{1}{2} + C_1$ or $C_1 = \frac{Wal}{2}$

$\therefore$ $EI\frac{dy}{dx} = -Wax + \frac{Wal}{2}$

Integrating again, we get

$$EIy = -Wa\frac{x^2}{2} + \frac{Wal}{2}x + C_2$$

When $x = a, y = 0$

$\therefore$ $0 = -\frac{Wa^3}{2} + \frac{Wa^2l}{2} + C_2$

or $C_2 = \frac{Wa^3}{2} - \frac{Wa^2l}{2}$

$\therefore$ $EIy = -\frac{Wax^2}{2} + \frac{Walx}{2} + \left[\frac{Wa^3}{2} - \frac{Wa^2l}{2}\right]$

or $y = \frac{Wa}{EI}\left[-\frac{x^2}{2} + \frac{lx}{2} + \frac{a^2}{2} - \frac{al}{2}\right]$

At mid – span, i, e., $x = l/2$

$$y = \frac{Wa}{EI}\left[-\frac{(l/2)^2}{2} + \frac{l\times(l/2)}{2} + \frac{a^2}{2} - \frac{al}{2}\right]$$

$$= \frac{Wa}{EI}\left[-\frac{l^2}{8} + \frac{a^2}{2} - \frac{al}{2}\right]$$

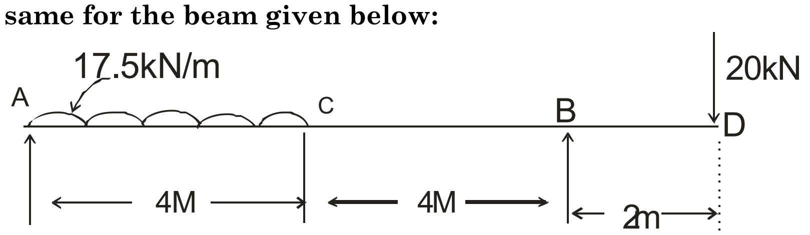
$$= \frac{1\times1000\times0.125}{208\times10^9\times8.59\times10^{-8}}\left[\frac{1.5^2}{8} + \frac{0.125^2}{2} - \frac{0.125\times1.5}{2}\right]$$

$$= 0.001366m = 1.366mm$$

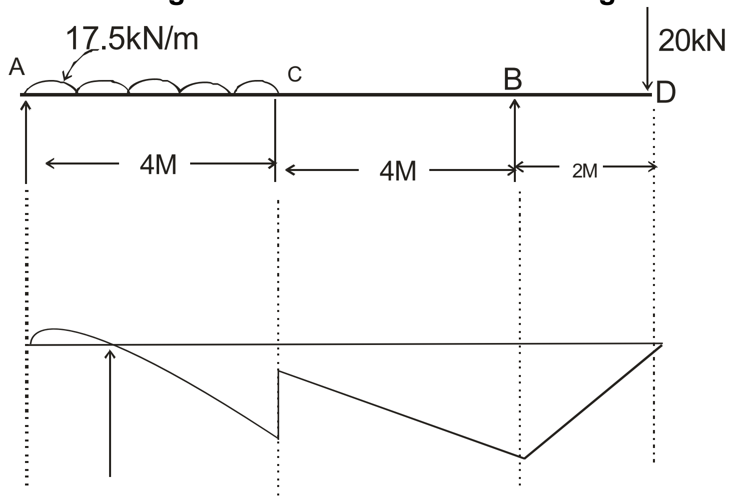
It will be in upward direction

Conventional Question IES-2001

Question: **What is meant by point of contraflexure or point of inflexion in a beam? Show the same for the beam given below:**



Answer: In a beam if the bending moment changes sign at a point, the point itself having zero bending moment, the beam changes curvature at this point of zero bending moment and this point is called the point of contra flexure.



BMD

From the bending moment diagram we have seen that it is between A & C.
[If marks are more we should calculate exact point.]

5. Deflection of Beam

Theory at a Glance (for IES, GATE, PSU)

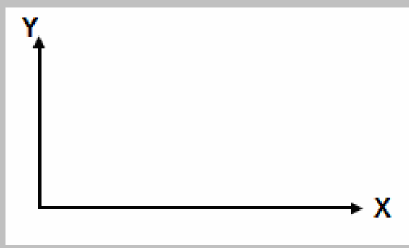
5.1 Introduction

- We know that the axis of a beam deflects from its initial position under action of applied forces.
- In this chapter we will learn how to determine the elastic deflections of a beam.

Selection of co-ordinate axes

We will not introduce any other co-ordinate system. We use general co-ordinate axis as shown in the figure. This system will be followed in deflection of beam and in shear force and bending moment diagram. Here downward direction will be negative i.e. negative Y-axis. Therefore downward deflection of the beam will be treated as negative.

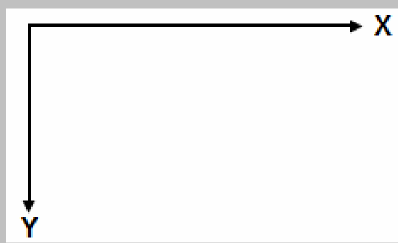
To determine the value of deflection of beam subjected to a given loading where we will use the formula, $EI\frac{d^2y}{dx^2} = M_x$.



We use above Co-ordinate system

Some books fix a co-ordinate axis as shown in the following figure. Here downward direction will be positive i.e. positive Y-axis. Therefore downward deflection of the beam will be treated as positive. As beam is generally deflected in downward directions and this co-ordinate system treats downward deflection is positive deflection.

To determine the value of deflection of beam subjected to a given loading where we will use the formula, $EI\frac{d^2y}{dx^2} = -M_x$.



Some books use above co-ordinate system

Why to calculate the deflections?

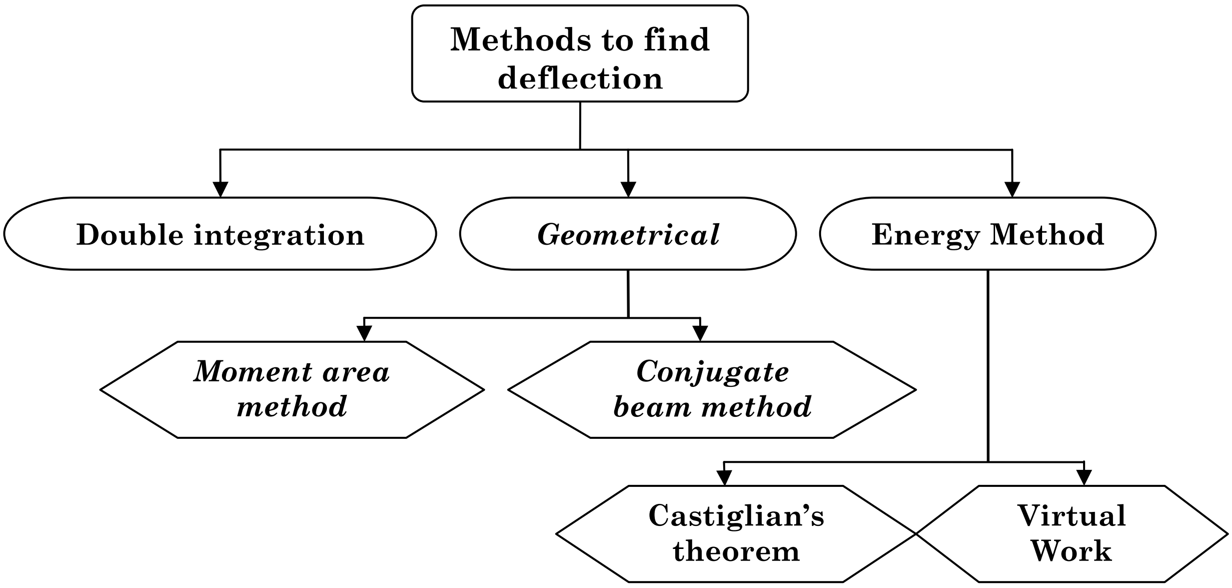
- To prevent cracking of attached brittle materials
- To make sure the structure not deflect severely and to “appear” safe for its occupants
- To help analyzing statically indeterminate structures

- Information on deformation characteristics of members is essential in the study of vibrations of machines

Several methods to compute deflections in beam

- Double integration method (*without* the use of singularity functions)
- Macaulay's Method (*with* the use of singularity functions)
- Moment area method
- Method of superposition
- Conjugate beam method
- Castigliano's theorem
- Work/Energy methods

Each of these methods has particular advantages or disadvantages.



Assumptions in Simple Bending Theory

- Beams are initially straight
- The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions
- The stress-strain relationship is linear and elastic
- Young's Modulus is the same in tension as in compression
- Sections are symmetrical about the plane of bending
- Sections which are plane before bending remain plane after bending

Non-Uniform Bending

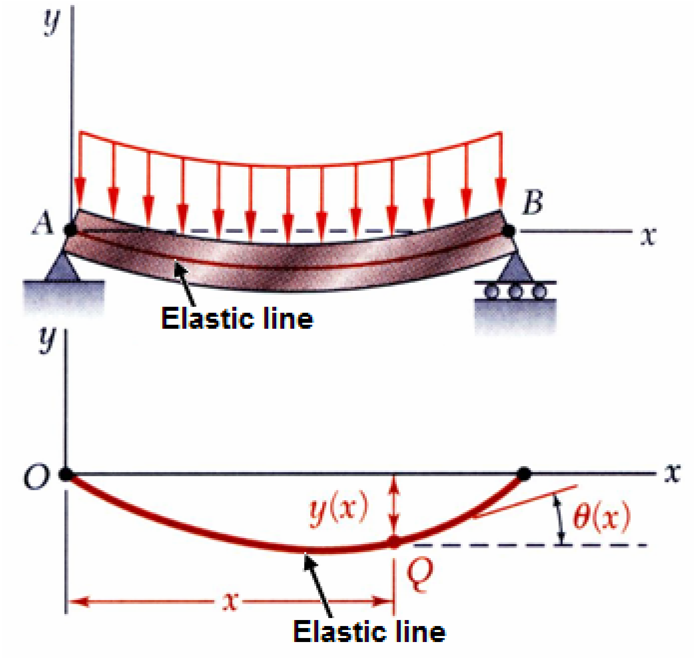
- In the case of non-uniform bending of a beam, where bending moment varies from section to section, there will be shear force at each cross section which will induce shearing stresses
- Also these shearing stresses cause warping (or out-of plane distortion) of the cross section so that plane cross sections do not remain plane even after bending

5.2 Elastic line or Elastic curve

We have to remember that the differential equation of the elastic line is

$$EI\frac{d^2y}{dx^2}=M_x$$

Proof: Consider the following simply supported beam with UDL over its length.



From elementary calculus we know that curvature of a line (at point Q in figure)

$$\frac{1}{R}=\frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}} \qquad \text{where } R=\text{radius of curvature}$$

For small deflection, $\frac{dy}{dx}\approx 0$

$$\text{or } \frac{1}{R}\approx\frac{d^2y}{dx^2}$$

Bending stress of the beam (at point Q)

$$\sigma_x = \frac{-(M_x).y}{EI}$$

From strain relation we get

$$\frac{1}{R} = -\frac{\varepsilon_x}{y} \text{ and } \varepsilon_x = \frac{\sigma_x}{E}$$

$$\therefore \quad \frac{1}{R} = \frac{M_x}{EI}$$

Therefore $\frac{d^2y}{dx^2} = \frac{M_x}{EI}$

or $EI\frac{d^2y}{dx^2} = M_x$

5.3 General expression

From the equation $EI\frac{d^2y}{dx^2} = M_x$ we may easily find out the following relations.

- $EI\frac{d^4y}{dx^4} = -\omega$ Shear force density (Load)
- $EI\frac{d^3y}{dx^3} = V_x$ Shear force
- $EI\frac{d^2y}{dx^2} = M_x$ Bending moment
- $\frac{dy}{dx} = \theta = \text{slope}$
- y = δ = Deflection, Displacement
- Flexural rigidity = *EI*

5.4 Double integration method (*without* the use of singularity functions)

- $V_x = \int -\omega dx$
- $M_x = \int V_x dx$
- $EI\frac{d^2y}{dx^2} = M_x$
- $\theta = Slope = \frac{1}{EI}\int M_x dx$
- $\delta = Deflection = \int \theta dx$

4-step procedure to solve deflection of beam problems by double integration method

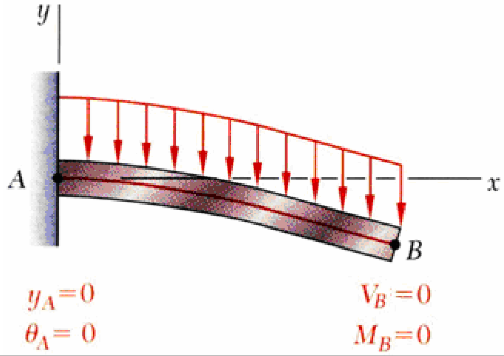
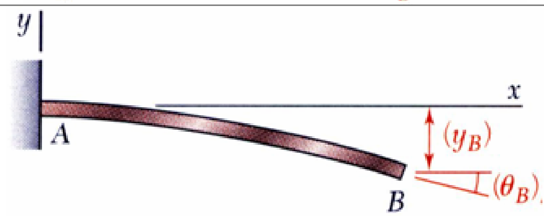
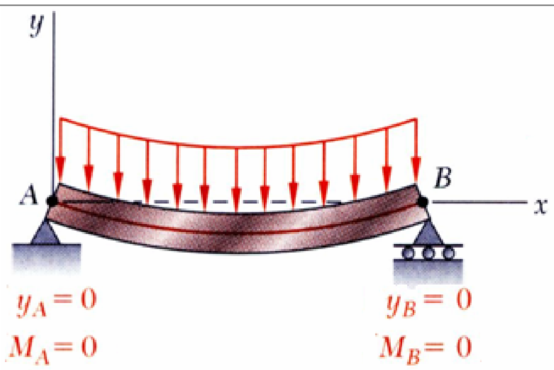
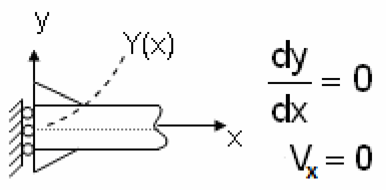
Step 1: Write down boundary conditions (Slope boundary conditions and displacement boundary conditions), analyze the problem to be solved

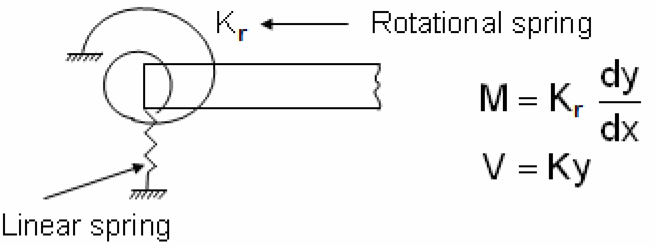
Step 2: Write governing equations for, $EI\dfrac{d^2y}{dx^2} = M_x$

Step 3: Solve governing equations by integration, results in expression with unknown integration constants

Step 4: Apply boundary conditions (determine integration constants)

Following table gives boundary conditions for different types of support.

Types of support and Boundary Conditions	Figure
Clamped or Built in support or Fixed end : (Point A) $Deflection, (y) = 0$ $Slope, (\theta) = 0$ $Moment, (M) \neq 0 \quad i.e.$ A finite value	
Free end: (Point B) $Deflection, (y) \neq 0 \quad i.e.$ A finite value $Slope, (\theta) \neq 0 \quad i.e.$ A finite value $Moment, (M) = 0$	
Roller (Point B) **or Pinned Support** (Point A) **or Hinged or Simply supported.** $Deflection, (y) = 0$ $Slope, (\theta) \neq 0 \quad i.e.$ A finite value $Moment, (M) = 0$	
End restrained against rotation but free to deflection $Deflection, (y) \neq 0 \quad i.e.$ A finite value $Slope, (\theta) = 0$ $Shear\ force, (V) = 0$	
Flexible support $Deflection, (y) \neq 0 \quad i.e.$ A finite value $Slope, (\theta) \neq 0 \quad i.e.$ A finite value	

$Moment, (M) = k_r \dfrac{dy}{dx}$ $Shear\ force, (V) = k.y$	

Using double integration method we will find the deflection and slope of the following loaded beams one by one.

(i) A Cantilever beam with point load at the free end.

(ii) A Cantilever beam with UDL (uniformly distributed load)

(iii) A Cantilever beam with an applied moment at free end.

(iv) A simply supported beam with a point load at its midpoint.

(v) A simply supported beam with a point load NOT at its midpoint.

(vi) A simply supported beam with UDL (Uniformly distributed load)

(vii) A simply supported beam with triangular distributed load (GVL) gradually varied load.

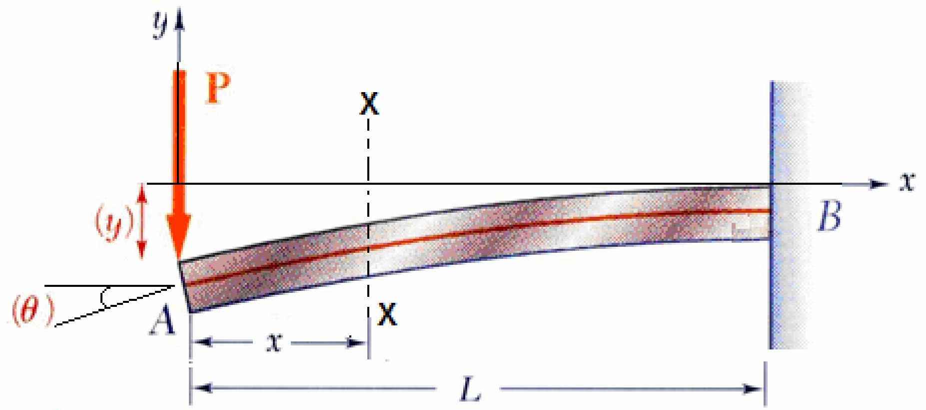
(viii) A simply supported beam with a moment at mid span.

(ix) A simply supported beam with a continuously distributed load the intensity of which at any point 'x' along the beam is $w_x = w \sin\left(\dfrac{\pi x}{L}\right)$

(i) A Cantilever beam with point load at the free end.

We will solve this problem by double integration method. For that at first we have to calculate (M_x).

Consider any section XX at a distance 'x' from free end which is left end as shown in figure.



$\therefore\ M_x = -P.x$

We know that differential equation of elastic line

$$EI \frac{d^2y}{dx^2} = M_x = -P.x$$

Integrating both side we get

$$\int EI \frac{d^2y}{dx^2} = -P\int x\,dx$$

$$\text{or } EI\frac{dy}{dx} = -P.\frac{x^2}{2} + A \qquad \text{.............(i)}$$

Again integrating both side we get

$$EI\int dy = \int\left(P\frac{x^2}{2} + A\right)dx$$

$$\text{or } EIy = -\frac{Px^3}{6} + Ax + B \qquad \text{..............(ii)}$$

Where A and B is integration constants.

Now apply boundary condition at fixed end which is at a distance x = L from free end and we also know that at fixed end

at $x = L,\quad y = 0$

at $x = L,\quad \frac{dy}{dx} = 0$

from equation (ii) $EIL = -\frac{PL^3}{6} + AL + B$ ……….(iii)

from equation (i) $EI.(0) = -\frac{PL^2}{2} + A$ …..(iv)

Solving (iii) & (iv) we get $A = \frac{PL^2}{2}$ and $B = -\frac{PL^3}{3}$

Therefore, $y = -\frac{Px^3}{6EI} + \frac{PL^2x}{2EI} - \frac{PL^3}{3EI}$

The slope as well as the deflection would be maximum at free end hence putting x = 0 we get

$y_{max} = -\frac{PL^3}{3EI}$ (Negative sign indicates the deflection is downward)

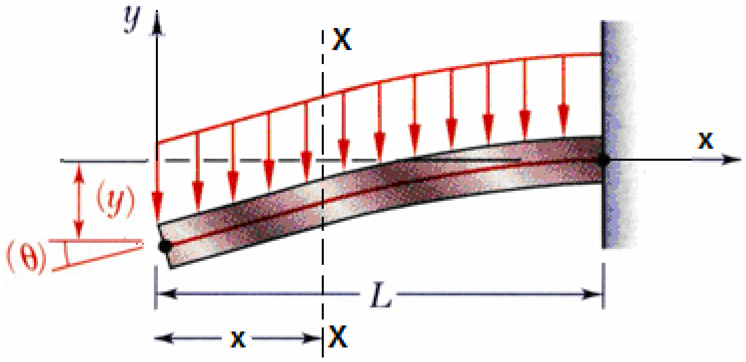
$(\text{Slope})_{max} = \theta_{max} = \frac{PL^2}{2EI}$

Remember for a cantilever beam with a point load at free end.

Downward deflection at free end, $(\delta) = \frac{PL^3}{3EI}$

And slope at free end, $(\theta) = \frac{PL^2}{2EI}$

(ii) A Cantilever beam with UDL (uniformly distributed load)



We will now solve this problem by double integration method, for that at first we have to calculate (M_x).

Consider any section XX at a distance 'x' from free end which is left end as shown in figure.

$$\therefore M_x = -(w.x).\frac{x}{2} = -\frac{wx^2}{2}$$

We know that differential equation of elastic line

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

Integrating both sides we get

$$\int EI\frac{d^2y}{dx^2} = \int -\frac{wx^2}{2}dx$$

$$or \quad EI\frac{dy}{dx} = -\frac{wx^3}{6} + A \quad ......(i)$$

Again integrating both side we get

$$EI\int dy = \int\left(-\frac{wx^3}{6} + A\right)dx$$

$$\text{or} \quad EIy = -\frac{wx^4}{24} + Ax + B.......(ii)$$

[where A and B are integration constants]

Now apply boundary condition at fixed end which is at a distance x = L from free end and we also know that at fixed end.

$$\text{at } x = L, \quad y = 0$$

$$\text{at } x = L, \quad \frac{dy}{dx} = 0$$

from equation (i) we get $EI \times (0) = \frac{-wL^3}{6} + A$ or $A = \frac{+wL^3}{6}$

from equation (ii) we get $EI.y = -\frac{wL^4}{24} + A.L + B$

$$\text{or} \quad B = -\frac{wL^4}{8}$$

The slope as well as the deflection would be maximum at the free end hence putting x = 0, we get

$$y_{max} = -\frac{wL^4}{8EI} \qquad \text{[Negative sign indicates the deflection is downward]}$$

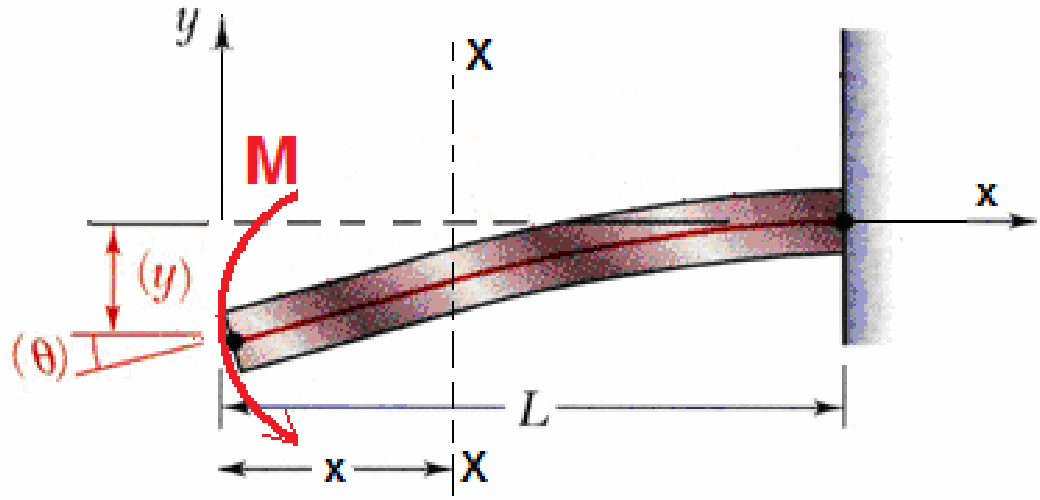
$$(\text{slope})_{max} = \theta_{max} = \frac{wL^3}{6EI}$$

Remember: For a cantilever beam with UDL over its whole length,

Maximum deflection at free end $\boxed{(\delta) = \dfrac{wL^4}{8EI}}$

Maximum slope, $\boxed{(\theta) = \dfrac{wL^3}{6EI}}$

(iii) A Cantilever beam of length 'L' with an applied moment 'M' at free end.



Consider a section XX at a distance 'x' from free end, the bending moment at section XX is

$(M_x) = -M$

We know that differential equation of elastic line

$$\text{or } EI\frac{d^2y}{dx^2} = -M$$

Integrating both side we get

$$\text{or } EI\int\frac{d^2y}{dx^2} = -\int M\,dx$$

$$\text{or } EI\frac{dy}{dx} = -Mx + A \;...(i)$$

Again integrating both side we get

$$EI\int dy = \int (M\,x + A)\,dx$$

or $EI\ y = -\dfrac{Mx^2}{2} + Ax + B$ ...(ii)

Where A and B are integration constants.

applying boundary conditions in equation (i) &(ii)

at $x = L,\ \dfrac{dy}{dx} = 0$ gives $A = ML$

at $x = L,\ y = 0$ gives $B = \dfrac{ML^2}{2} - ML^2 = -\dfrac{ML^2}{2}$

Therefore deflection equation is $y = -\dfrac{Mx^2}{2EI} + \dfrac{MLx}{EI} - \dfrac{ML^2}{2EI}$

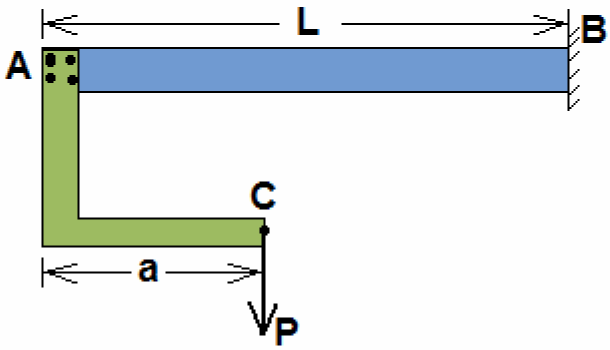
Which is the equation of elastic curve.

$\therefore$ Maximum deflection at free end $$(\delta) = \frac{ML^2}{2EI}$$ (It is downward)

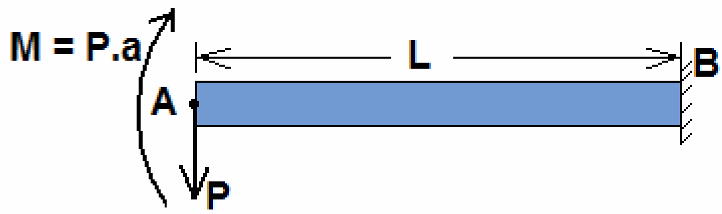
$\therefore$ Maximum slope at free end $$(\theta) = \frac{ML}{EI}$$

Let us take a funny example: A cantilever beam AB of length 'L' and uniform flexural rigidity EI has a bracket BA (attached to its free end. A vertical downward force P is applied to free end C of the bracket. Find the ratio a/L required in order that the deflection of point A is zero.

[ISRO – 2008]



We may consider this force 'P' and a moment (P.a) act on free end A of the cantilever beam.



Due to point load 'P' at free end 'A' downward deflection $\left(\delta\right) = \dfrac{PL^3}{3EI}$

Due to moment M = P.a at free end 'A' upward deflection $\left(\delta\right) = \dfrac{ML^2}{2EI} = \dfrac{(P.a)L^2}{2EI}$

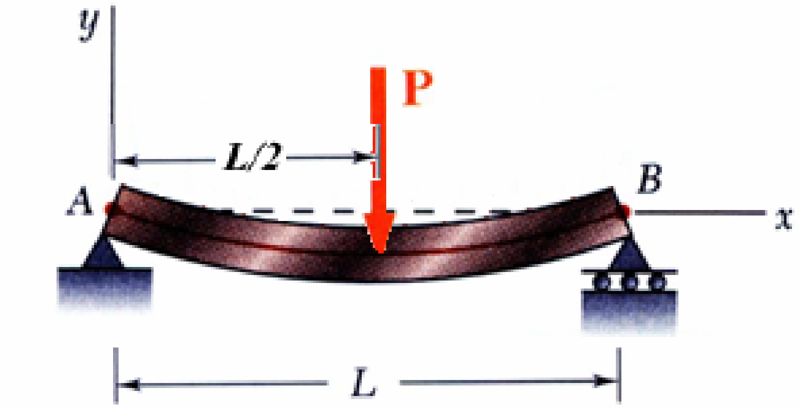
For zero deflection of free end A

$$\frac{PL^3}{3EI} = \frac{(P.a)L^2}{2EI}$$

$$\text{or } \frac{a}{L} = \frac{2}{3}$$

(iv) A simply supported beam with a point load P at its midpoint.

A simply supported beam AB carries a concentrated load P at its midpoint as shown in the figure.



We want to locate the point of maximum deflection on the elastic curve and find its value.

In the region 0 < x < L/2

Bending moment at any point x (According to the shown co-ordinate system)

$$M_x = \left(\frac{P}{2}\right).x$$

and **In the region L/2 < x < L**

$$M_x = \frac{P}{2}\left(x - L/2\right)$$

We know that differential equation of elastic line

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}.x \qquad \left(\text{In the region } 0 < x < L/2\right)$$

Integrating both side we get

$$\text{or EI} \int\frac{d^2y}{dx^2} = \int\frac{P}{2}x\,dx$$

$$\text{or EI}\frac{dy}{dx} = \frac{P}{2}.\frac{x^2}{2} + A \quad \text{(i)}$$

Again integrating both side we get

$$EI \int dy = \int \left(\frac{P}{4}x^2 + A \right) dx$$

$$\text{or } EI\ y = \frac{Px^3}{12} + Ax + B \text{ (ii)}$$

[Where A and B are integrating constants]

Now applying boundary conditions to equation (i) and (ii) we get

at $x = 0, \quad y = 0$

at $x = L/2, \ \frac{dy}{dx} = 0$

$A = -\frac{PL^2}{16}$ and $B = 0$

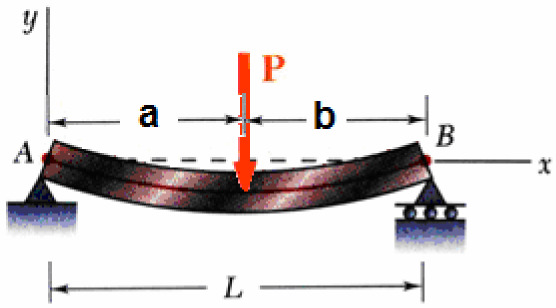
$\therefore$ Equation of elastic line, $y = \frac{Px^3}{12} - \frac{PL^{12}}{16}x$

Maximum deflection at mid span (x = L/2) $\boxed{(\delta) = \frac{PL^3}{48EI}}$

and maximum slope at each end $\boxed{(\theta) = \frac{PL^2}{16EI}}$

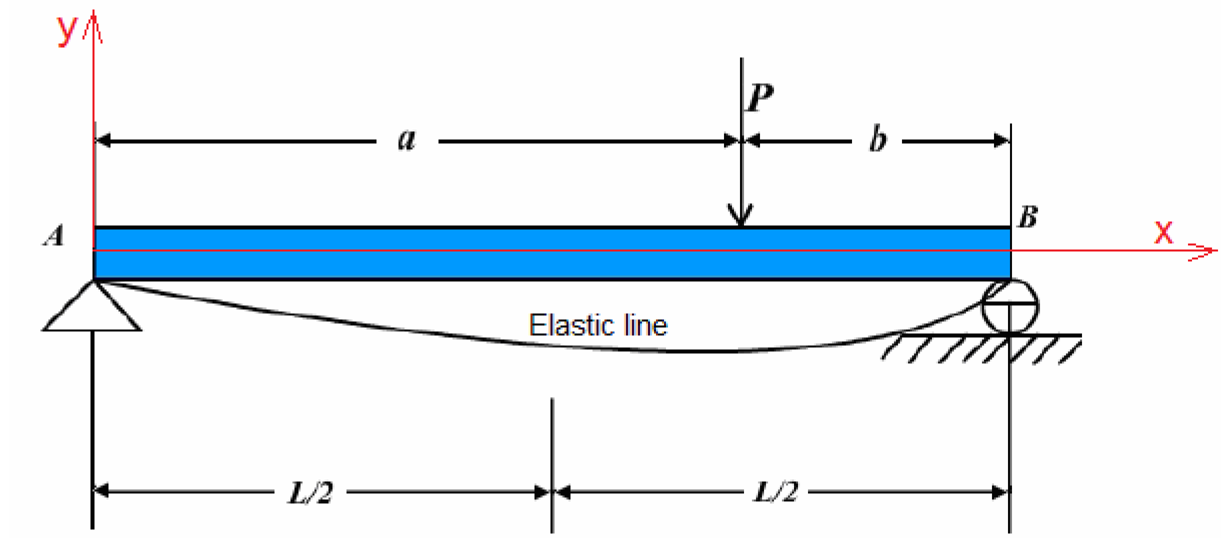
(v) A simply supported beam with a point load 'P' NOT at its midpoint.

A simply supported beam AB carries a concentrated load P as shown in the figure.



We have to locate the point of maximum deflection on the elastic curve and find the value of this deflection.

Taking co-ordinate axes x and y as shown below



For the bending moment we have

In the region $0 \le x \le a$, $\qquad M_x = \left(\frac{P.a}{L}\right).x$

And, In the region $a \le x \le L$, $\qquad M_x = -\frac{P.a}{L}(L - x)$

So we obtain two differential equation for the elastic curve.

$$EI\frac{d^2y}{dx^2} = \frac{P.a}{L}.x \qquad \text{for } 0 \le x \le a$$

$$\text{and } EI\frac{d^2y}{dx^2} = -\frac{P.a}{L}.(L - x) \qquad \text{for } a \le x \le L$$

Successive integration of these equations gives

$$EI\frac{dy}{dx} = \frac{P.a}{L}.\frac{x^2}{2} + A_1 \qquad \text{......(i)} \qquad \text{for } o \le x \le a$$

$$EI\frac{dy}{dx} = P.a\,x - \frac{P.a}{L}x^2 + A_2 \qquad \text{......(ii)} \qquad \text{for } a \le x \le L$$

$$EI\,y = \frac{P.a}{L}.\frac{x^3}{6} + A_1x + B_1 \qquad \text{......(iii)} \qquad \text{for } 0 \le x \le a$$

$$EI\,y = P.a\frac{x^2}{2} - \frac{P.a}{L}.\frac{x^3}{6} + A_2\,x + B_2 \qquad \text{.....(iv)} \qquad \text{for } a \le x \le L$$

Where A_1, A_2, B_1, B_2 are constants of Integration.

Now we have to use Boundary conditions for finding constants:

BCS (a) at $x = 0$, $y = 0$

(b) at $x = L$, $y = 0$

(c) at $x = a$, $\left(\frac{dy}{dx}\right)$ = Same for equation (i) & (ii)

(d) at $x = a$, y = same from equation (iii) & (iv)

We get $\qquad A_1 = \frac{Pb}{6L}\left(L^2 - b^2\right); \qquad A_2 = \frac{P.a}{6L}\left(2L^2 + a^2\right)$

$$\text{and } B_1 = 0; \qquad B_2 = Pa^3 / 6EI$$

Therefore we get two equations of elastic curve

$$EI\ y = -\frac{Pbx}{6L}\left(L^2 - b^2 - x^2\right) \qquad \text{..... (v)} \qquad \text{for } 0 \le x \le a$$

$$EI\ y = \frac{Pb}{6L}\left[\left(\frac{L}{b}\right)(x-a)^3 + \left(L^2 - b^2\right)x - x^3\right] \text{. ...(vi)} \qquad \text{for } a \le x \le L$$

For a > b, the maximum deflection will occur in the left portion of the span, to which equation (v) applies. Setting the derivative of this expression equal to zero gives

$$x = \sqrt{\frac{a(a+2b)}{3}} = \sqrt{\frac{(L-b)(L+b)}{3}} = \sqrt{\frac{L^2 - b^2}{3}}$$

at that point a horizontal tangent and hence the point of maximum deflection substituting this value of x into equation (v), we find, $\mathbf{y_{max} = \dfrac{P.b(L^2 - b^2)^{3/2}}{9\sqrt{3}.\ EIL}}$

Case –I: if a = b = L/2 then

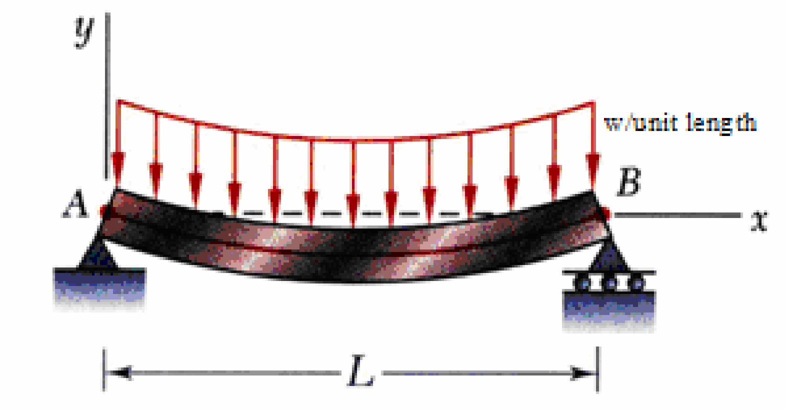
Maximum deflection will be at $x = \sqrt{\dfrac{L^2 - (L/2)^2}{3}} = L/2$

i.e. at mid point

and $y_{max} = (\delta) = \dfrac{P.(L/2) \times \left\{L^2 - (L/2)^2\right\}^{3/2}}{9\sqrt{3}\,EIL} = \dfrac{PL^3}{48EI}$

(vi) A simply supported beam with UDL (Uniformly distributed load)

A simply supported beam AB carries a uniformly distributed load (UDL) of intensity w/unit length over its whole span L as shown in figure. We want to develop the equation of the elastic curve and find the maximum deflection δ at the middle of the span.



Taking co-ordinate axes x and y as shown, we have for the bending moment at any point x

$$M_x = \frac{wL}{2}.x - w.\frac{x^2}{2}$$

Then the differential equation of deflection becomes

$$EI\ \frac{d^2y}{dx^2} = M_x = \frac{wL}{2}.x - w.\frac{x^2}{2}$$

Integrating both sides we get

$$EI\ \frac{dy}{dx} = \frac{wL}{2}.\frac{x^2}{2} - \frac{w}{2}.\frac{x^3}{3} + A \qquad \text{.....(i)}$$

Again Integrating both side we get

$$EI\,y = \frac{wL}{2}.\frac{x^3}{6} - \frac{w}{2}.\frac{x^4}{12} + Ax + B \qquad .....(ii)$$

Where A and B are integration constants. To evaluate these constants we have to use boundary conditions.

at $x = 0, y = 0$ gives $B = 0$

at $x = L/2$, $\frac{dy}{dx} = 0$ gives $A = -\frac{wL^3}{24}$

Therefore the equation of the elastic curve

$$y = \frac{wL}{12EI}.x^3 - \frac{w}{24EI}.x^4 - \frac{wL^3}{12EI}.x = \frac{wx}{24EI}\left[L^3 - 2L.x^2 + x^3\right]$$

The maximum deflection at the mid-span, we have to put x = L/2 in the equation and obtain

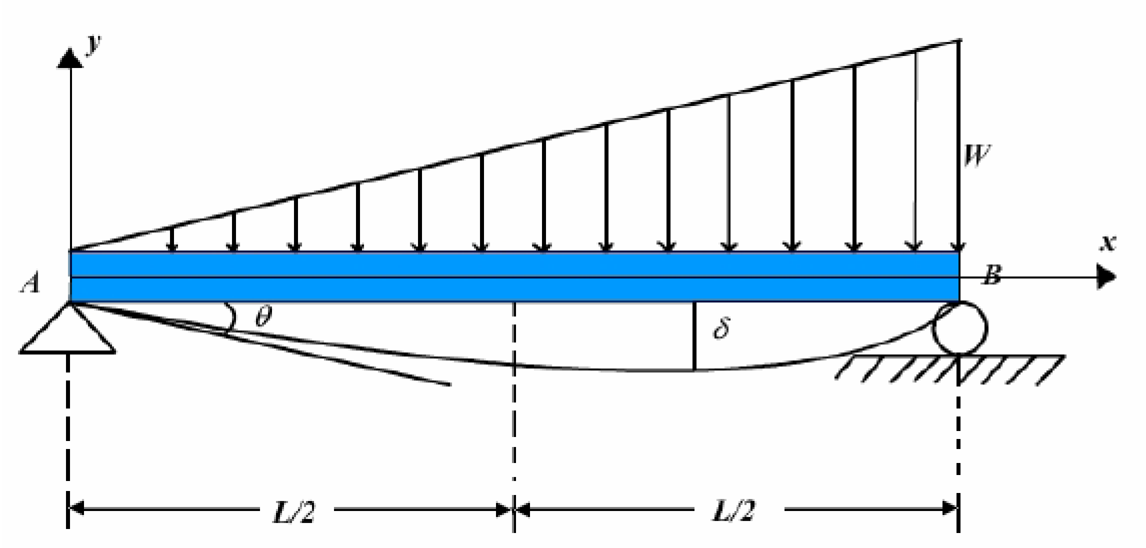
Maximum deflection at mid-span, $$(\delta) = \frac{5wL^4}{384EI}$$ (It is downward)

And Maximum slope $\theta_A = \theta_B$ at the left end A and at the right end b is same putting x = 0 or x = L

Therefore we get Maximum slope $$(\theta) = \frac{wL^3}{24EI}$$

(vii) A simply supported beam with triangular distributed load (GVL) gradually varied load.

A simply supported beam carries a triangular distributed load (GVL) as shown in figure below. We have to find equation of elastic curve and find maximum deflection (δ).



In this (GVL) condition, we get

$$EI\frac{d^4y}{dx^4} = load = -\frac{w}{L}.x \quad .....(i)$$

Separating variables and integrating we get

$$EI\frac{d^3y}{dx^3} = (V_x) = -\frac{wx^2}{2L} + A \quad .....(ii)$$

Again integrating thrice we get

$$EI\frac{d^2y}{dx^2} = M_x = -\frac{wx^3}{6L} + Ax + B \quad .....(iii)$$

$$EI\frac{dy}{dx} = -\frac{wx^4}{24L} + \frac{Ax^2}{2} + Bx + C \quad .....(iv)$$

$$EI\,y = -\frac{wx^5}{120L} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \quad .....(v)$$

Where A, B, C and D are integration constant.

Boundary conditions at x = 0, $M_x = 0$, y = 0

at x = L, $M_x = 0$, y = 0 gives

$A = \frac{wL}{6}$, B = 0, $C = -\frac{7wL^3}{360}$, D = 0

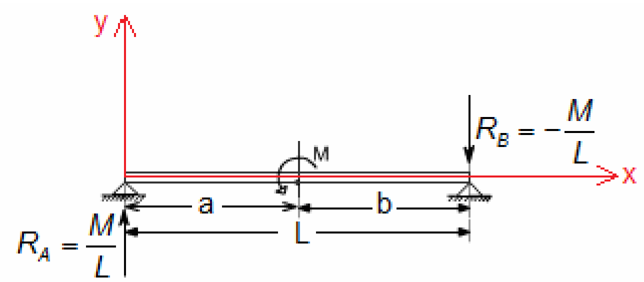
Therefore $y = -\frac{wx}{360EIL}\{7L^4 - 10L^2x^2 + 3x^4\}$ *(negative sign indicates downward deflection)*

To find maximum deflection δ, we have $\frac{dy}{dx} = 0$

And it gives x = 0.519 L and maximum deflection $(\delta) = 0.00652\frac{wL^4}{EI}$

(viii) A simply supported beam with a moment at mid-span

A simply supported beam AB is acted upon by a couple M applied at an intermediate point distance 'a' from the equation of elastic curve and deflection at point where the moment acted.



Considering equilibrium we get $R_A = \frac{M}{L}$ and $R_B = -\frac{M}{L}$

Taking co-ordinate axes x and y as shown, we have for bending moment

In the region $0 \le x \le a$, $M_x = \frac{M}{L}.x$

In the region $a \le x \le L$, $M_x = \frac{M}{L}x - M$

So we obtain the difference equation for the elastic curve

$$EI\frac{d^2y}{dx^2} = \frac{M}{L}.x \qquad \text{for } 0 \le x \le a$$

$$\text{and } EI\frac{d^2y}{dx^2} = \frac{M}{L}.x - M \qquad \text{for } a \le x \le L$$

Successive integration of these equation gives

$$EI\frac{dy}{dx} = \frac{M}{L}.\frac{x^2}{2} + A_1 \qquad ....(i) \qquad \text{for } 0 \le x \le a$$

$$EI\frac{dy}{dx} = \frac{M}{L} = \frac{x^2}{2} \text{- } Mx + A_2 \qquad .....(ii) \qquad \text{for } a \le x \le L$$

$$\text{and } EI\ y = \frac{M}{L}.\frac{x^3}{\sigma} + A_1x + B_1 \qquad ......(iii) \qquad \text{for } 0 \le x \le a$$

$$EI\ y = \frac{M}{L}\ \frac{x^3}{\sigma} - \frac{Mx^2}{2} + A_2x + B_2 \qquad ......(iv) \qquad \text{for } a \le x \le L$$

Where A_1, A_2, B_1 and B_2 are integration constants.

To finding these constants boundary conditions

(a) at $x = 0,\ y = 0$

(b) at $x = L,\ y = 0$

(c) at $x = a$, $\left(\frac{dy}{dx}\right)$ = same form equation (i) & (ii)

(d) at $x = a$, y = same form equation (iii) & (iv)

$$A_1 = -M.a + \frac{ML}{3} + \frac{Ma^2}{2L},\quad A_2 = \frac{ML}{3} + \frac{Ma^2}{2L}$$

$$B_1 = 0, \qquad B_2 = \frac{Ma^2}{2}$$

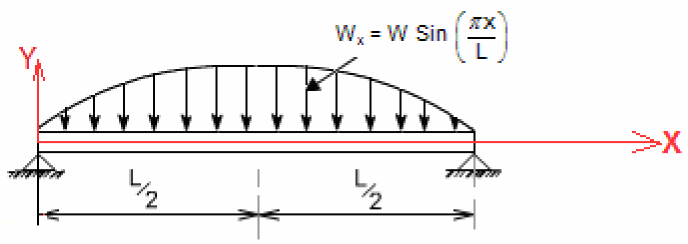
With this value we get the equation of elastic curve

$$y = -\frac{Mx}{6L}\left\{6aL - 3a^2 - x^2 - 2L^2\right\} \qquad \text{for } 0 \le x \le a$$

$\therefore$ deflection of x = a,

$$y = \frac{Ma}{3EIL}\left\{3aL - 2a^2 - L^2\right\}$$

(ix) A simply supported beam with a continuously distributed load the intensity of which at any point 'x' along the beam is $w_x = w\sin\left(\frac{\pi x}{L}\right)$



At first we have to find out the bending moment at any point 'x' according to the shown co-ordinate system.

We know that

$$\frac{d(V_x)}{dx} = -w\sin\left(\frac{\pi x}{L}\right)$$

Integrating both sides we get

$$\int d(V_x) = -\int w\ \sin\left(\frac{\pi x}{L}\right)dx + A$$

$$\text{or } V_x = +\frac{wL}{\pi}.\cos\left(\frac{\pi x}{L}\right) + A$$

and we also know that

$$\frac{d(M_x)}{dx} = V_x = \frac{wL}{\pi}\cos\left(\frac{\pi x}{L}\right) + A$$

Again integrating both sides we get

$$\int d(M_x) = \int \left\{\frac{wL}{\pi}\cos\left(\frac{\pi x}{L}\right) + A\right\}dx$$

$$\text{or } M_x = \frac{wL^2}{\pi^2}\sin\left(\frac{\pi x}{L}\right) + Ax + B$$

Where A and B are integration constants, to find out the values of A and B. We have to use boundary conditions

at $x = 0$, $M_x = 0$

and at $x = L$, $M_x = 0$

From these we get $A = B = 0$. Therefore $M_x = \frac{wL^2}{\pi^2}\sin\left(\frac{\pi x}{L}\right)$

So the differential equation of elastic curve

$$EI\ \frac{d^2y}{dx^2} = M_x = \frac{wL^2}{\pi^2}\sin\left(\frac{\pi x}{L}\right)$$

Successive integration gives

$$EI\frac{dy}{dx} = -\frac{wL^3}{\pi^3}\cos\left(\frac{\pi x}{L}\right) + C \qquad ....(i)$$

$$EI\,y = -\frac{wL^4}{\pi^4}\sin\left(\frac{\pi x}{L}\right) + Cx + D \qquad .....(ii)$$

Where C and D are integration constants, to find out C and D we have to use boundary conditions

at $x = 0$, $y = 0$

at $x = L$, $y = 0$

and that give $C = D = 0$

Therefore slope equation $EI\frac{dy}{dx} = -\frac{wL^3}{\pi^3}\cos\left(\frac{\pi x}{L}\right)$

and Equation of elastic curve $y = -\frac{wL^4}{\pi^4 EI}\sin\left(\frac{\pi x}{L}\right)$

(-ive sign indicates deflection is downward)

Deflection will be maximum if $\sin\left(\frac{\pi x}{L}\right)$ is maximum

$$\sin\left(\frac{\pi x}{L}\right) = 1 \quad \text{or} \quad x = L/2$$

and Maximum downward deflection $(\delta) = \dfrac{WL^4}{\pi^4 EI}$ (downward).

5.5 Macaulay's Method (Use of singularity function)

- When the beam is subjected to point loads (but several loads) this is very convenient method for determining the deflection of the beam.
- In this method we will write single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.
- After integrating this equation we will find the integration constants which are valid for entire length of the beam. This method is known as ***method of singularity constant.***

Procedure to solve the problem by Macaulay's method

Step – I: Calculate all reactions and moments

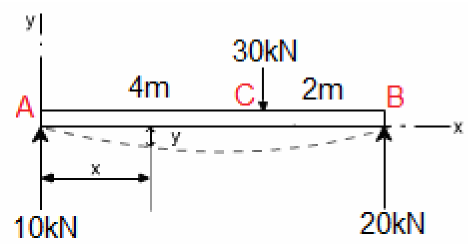
Step – II: Write down the moment equation which is valid for all values of x. This must contain brackets.

Step – III: Integrate the moment equation by a typical manner. Integration of (x-a) will be $\dfrac{(x\text{-}a)^2}{2}$ not $\left(\dfrac{x^2}{2} - ax\right)$ and integration of $(x\text{-}a)^2$ will be $\dfrac{(x\text{-}a)^3}{3}$ so on.

Step – IV: After first integration write the first integration constant (A) after first terms and after second time integration write the second integration constant (B) after A.x . Constant A and B are valid for all values of x.

Step – V: Using Boundary condition find A and B at a point x = p if any term in Macaulay's method, (x-a) is negative (-ive) the term will be neglected.

(i) Let us take an example: A simply supported beam AB length 6m with a point load of 30 kN is applied at a distance 4m from left end A. Determine the equations of the elastic curve between each change of load point and the maximum deflection of the beam.



Answer: We solve this problem using Macaulay's method, for that first writes the general momentum equation for the last portion of beam BC of the loaded beam.

$$EI\,\frac{d^2y}{dx^2} = M_x = 10x \;\Big|{-30(x-4)}\Big|\; N.m \qquad ....(i)$$

By successive integration of this equation (using Macaulay's integration rule

e.g $\int (x-a)\,dx = \frac{(x-a)^2}{2}$)

We get

$$EI\,\frac{dy}{dx} = 5x^2 + A \;\Big|\, -15(x\text{-}4)^2 \,\Big| \quad N.m^2 \qquad .....(ii)$$

$$\text{and}\;\; EI\,y = \frac{5}{3}x^3 + Ax + B \;\Big|\, -5\,(x-4)^3 \,\Big|\, N.m^3 \qquad .....(iii)$$

Where A and B are two integration constants. To evaluate its value we have to use following boundary conditions.

at x = 0, y = 0

and at x = 6m, y = 0

Note: When we put x = 0, x - 4 is negativre (–ive) and this term will ***not*** be considered for x = 0 , so our equation will be EI y = $\frac{5}{3}x^3 + Ax + B$, and at x = 0 , y = 0 gives B = 0

But when we put x = 6, x-4 is positive (+ive) and this term will be considered for x = 6, y = 0 so our equation will be EI y = $\frac{5}{3}x^3$ + Ax + 0 – 5 (x – 4)^3

This gives

$$EI\,.(0) = \frac{5}{3}.6^3 + A.6 + 0 - 5(6-4)^3$$

or A = - 53

So our slope and deflection equation will be

$$EI\,\frac{dy}{dx} = 5x^2 - 53 \;\Big|\, -15(x-4)^2 \,\Big|$$

$$\text{and } EI\,y = \frac{5}{3}x^3 - 53x + 0 \;\Big|\, -5(x-4)^3 \,\Big|$$

Now we have two equations for entire section of the beam and we have to understand how we use these equations. Here if x < 4 then x – 4 is negative so this term will be deleted. That so why in the region $0 \le x \le 4m$ we will neglect (x – 4) term and our slope and deflection equation will be

$$EI\,\frac{dy}{dx} = 5x^2 - 53$$

$$\text{and} \qquad EI\,y = \frac{5}{3}x^3 - 53x$$

But in the region $4m < x \le 6m$, (x – 4) is positive so we include this term and our slope and deflection equation will be

$$EI\,\frac{dy}{dx} = 5x^2 - 53 - 15(x-4)^2$$

$$EI\,y = \frac{5}{3}x^3 - 53x - 5(x-4)^3$$

Now we have to find out maximum deflection, but we don't know at what value of 'x' it will be maximum. For this assuming the value of 'x' will be in the region $0 \le x \le 4m$.

Deflection (y) will be maximum for that $\frac{dy}{dx} = 0$ or $5x^2 - 53 = 0$ or x = 3.25 m as our calculated x is in the region $0 \le x \le 4m$; at x = 3.25 m deflection will be maximum

or $\quad EI\, y_{max} = \frac{5}{3} \times 3.25^3 - 53 \times 3.25$

or $\quad y_{max} = -\frac{115}{EI}$ (-ive sign indicates downward deflection)

But if you have any doubt that Maximum deflection may be in the range of $4 < x \le 6m$, use $EIy = 5x^2 - 53x - 5(x-4)^3$ and find out x. The value of x will be absurd that indicates the maximum deflection will not occur in the region $4 < x \le 6m$.

Deflection (y) will be maximum for that $\frac{dy}{dx} = 0$

or $\quad 5x^2 - 53 - 15(x-4)^2 = 0$

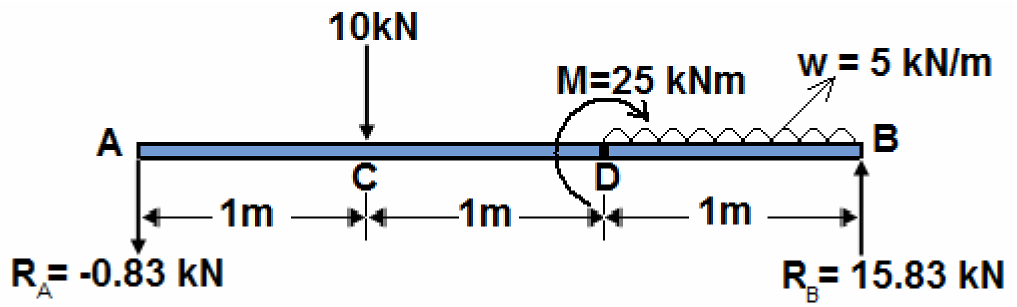
or $\quad 10x^2 - 120x + 293 = 0$

or $\quad x = 3.41$ m or 8.6 m

Both the value fall outside the region $4 < x \le 6m$ and in this region $4 < x \le 6m$ and in this region maximum deflection will not occur.

(ii) Now take an example where Point load, UDL and Moment applied simultaneously in a beam:

Let us consider a simply supported beam AB (see Figure) of length 3m is subjected to a point load 10 kN, UDL = 5 kN/m and a bending moment M = 25 kNm. Find the deflection of the beam at point D if flexural rigidity (EI) = 50 KNm².



Answer: Considering equilibrium

$$\sum M_A = 0 \text{ gives}$$

$$-10 \times 1 - 25 - (5 \times 1) \times (1 + 1 + 1/2) + R_B \times 3 = 0$$

$$\text{or } R_B = 15.83\,kN$$

$$R_A + R_B = 10 + 5 \times 1 \text{ gives } R_A = -0.83\,kN$$

We solve this problem using Macaulay's method, for that first writing the general momentum equation for the last portion of beam, DB of the loaded beam.

$$EI\frac{d^2y}{dx^2} = M_x = -0.83x \;\; \Big| -10(x-1) \Big| + 25(x-2)^0 \Big| - \frac{5(x-2)^2}{2} \Big|$$

By successive integration of this equation (using Macaulay's integration rule

e.g $\int (x-a)\,dx = \frac{(x-a)^2}{2}$)

We get

$$EI\ \frac{dy}{dx} = -\frac{0.83}{2}.x^2 + A\ \left|-5(x-1)^2\right| + 25(x-2)\left| -\frac{5}{6}(x-2)^3\right|$$

$$\text{and } EIy = -\frac{0.83}{6}x^3 + Ax + B\ \left|-\frac{5}{3}(x-1)^3\right| + \frac{25}{2}(x-2)^2\left| -\frac{5}{24}(x-2)^4\right|$$

Where A and B are integration constant we have to use following boundary conditions to find out A & B.

at x = 0, y = 0

at x = 3m, y = 0

Therefore B = 0

$$\text{and } 0 = -\frac{0.83}{6}\times 3^3 + A\times 3 + 0\left|-\frac{5}{3}\times 2^3\right| + 12.5\times 1^2\left| -\frac{5}{24}\times 1^4\right|$$

or A = 1.93

$$EIy = -0.138x^3 + 1.93x\ \left|-1.67(x-1)^3\right|\ +12.5(x-2)^2\left|-0.21(x-2)^4\right|$$

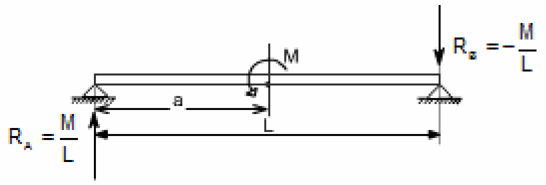
Deflextion at point D at x = 2m

$$EIy_D = -0.138\times 2^3 + 1.93\times 2 - 1.67\times 1^3 = -8.85$$

$$\text{or } y_D = -\frac{8.85}{EI} = -\frac{8.85}{50\times 10^3}\text{m} \ (-\text{ive sign indicates deflection downward})$$

$$= 0.177\text{mm}(\text{downward}).$$

(iii) A simply supported beam with a couple M at a distance 'a' from left end



If a couple acts we have to take the distance in the bracket and this should be raised to the power zero. i.e. $M(x-a)^0$. Power is zero because $(x-a)^0 = 1$ and unit of $M(x-a)^0 = M$ but we introduced the distance which is needed for Macaulay's method.

$$EI\ \frac{d^2y}{dx^2} = M = R_A x - M(x\text{-}a)^0$$

Successive integration gives

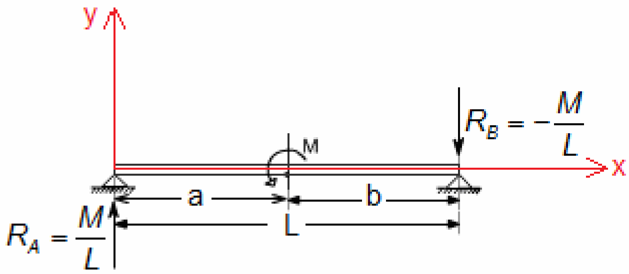
$$EI\ \frac{dy}{dx} = \frac{M}{L}.\frac{x^2}{2} + A - M(x\text{-}a)^1$$

$$EI\ y = \frac{M}{6L}x^3 + Ax + B - \frac{M(x\text{-}a)^2}{2}$$

Where A and B are integration constants, we have to use boundary conditions to find out A & B.

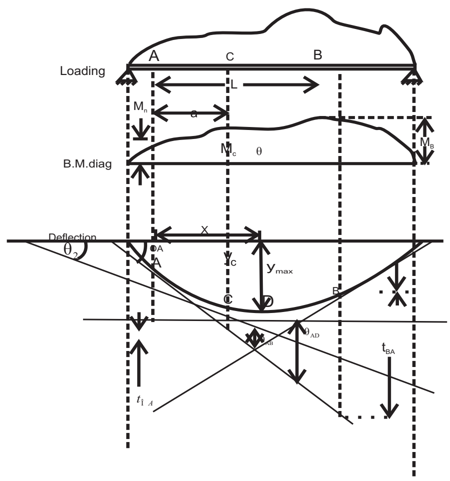
at x = 0, y = 0 gives B = 0

at x = L, y = 0 gives $A = \frac{M(L\text{-}a)^2}{2L} - \frac{ML}{6}$



8. Moment area method

- This method is used generally to obtain displacement and rotation at a single point on a beam.
- The moment area method is convenient in case of beams acted upon with point loads in which case bending moment area consist of triangle and rectangles.



- ***Angle between the tangents drawn at 2 points A&B on the elastic line,*** θ_{AB}

$$\theta_{AB} = \frac{1}{EI} \times \text{Area of the bending moment diagram between A\&B}$$

$$\text{i.e. slope } \boxed{\theta_{AB} = \frac{A_{B.M.}}{EI}}$$

- ***Deflection of B related to 'A'***

$$y_{BA} = \textbf{Moment of } \frac{M}{EI} \text{ diagram between B\&A taking about B (or w.r.t. } \textbf{B)}$$

$$\text{i.e. deflection } \boxed{y_{BA} = \frac{A_{B.M} \times \bar{x}}{EI}}$$

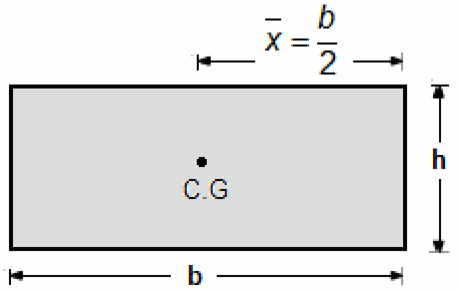
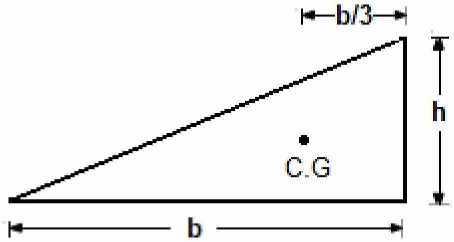
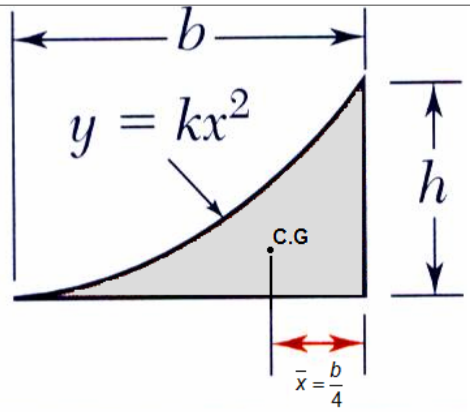
Important Note

If A_1 = Area of shear force (SF) diagram

A_2 = Area of bending moment (BM) diagram,

Then, Change of slope over any portion of the loaded beam = $\dfrac{A_1 \times A_2}{EI}$

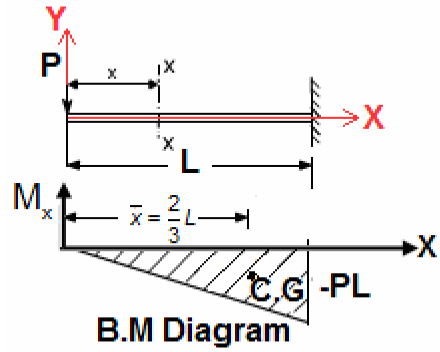
Some typical bending moment diagram and their area (A) and distance of C.G from one edge $\left(\bar{x}\right)$ is shown in the following table. [Note the distance will be different from other end]

Shape	BM Diagram	Area	Distance from C.G
1. Rectangle		$A = bh$	$\bar{x} = \frac{b}{2}$
2. Triangle			$\bar{x} = \frac{b}{3}$
3. Parabola			$\bar{x} = \frac{b}{4}$
4. Parabola			
5.Cubic Parabola			
6. $y = k x^n$			
7. Sine curve			

Determination of Maximum slope and deflection by Moment Area- Method

(i) A Cantilever beam with a point load at free end

Area of BM (Bending moment diagram)



$$(A) = \frac{1}{2} \times L \times PL = \frac{PL^2}{2}$$

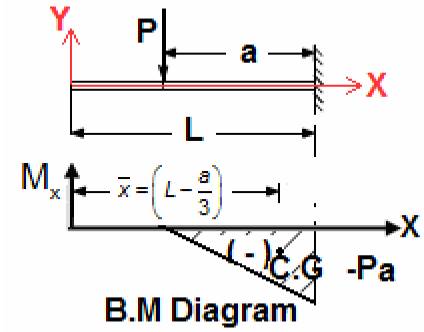
Therefore

$$\text{Maximum slope}(\theta) = \frac{A}{EI} = \frac{PL^2}{2EI} \qquad \text{(at free end)}$$

$$\text{Maximum deflection } (\delta) = \frac{A\bar{x}}{EI}$$

$$= \frac{\left(\frac{PL^2}{2}\right) \times \left(\frac{2}{3}L\right)}{EI} = \frac{PL^3}{3EI} \qquad \text{(at free end)}$$

(ii) A cantilever beam with a point load *not* at free end



$$\text{Area of BM diagram } (A) = \frac{1}{2} \times a \times Pa = \frac{Pa^2}{2}$$

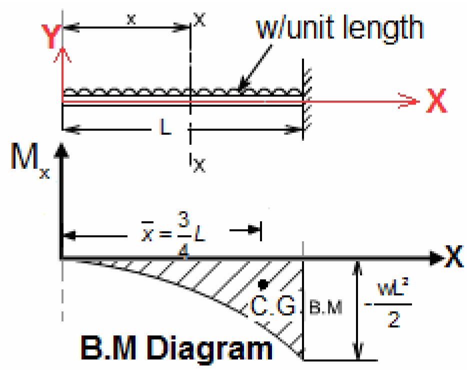
Therefore

$$\text{Maximum slope}(\theta) = \frac{A}{EI} = \frac{Pa^2}{2EI} \qquad \text{(at free end)}$$

$$\text{Maximum deflection } (\delta) = \frac{A\bar{x}}{EI}$$

$$= \frac{\left(\frac{Pa^2}{2}\right) \times \left(L - \frac{a}{3}\right)}{EI} = \frac{Pa^2}{2EI} \cdot \left(L - \frac{a}{3}\right) \qquad \text{(at free end)}$$

(iii) A cantilever beam with UDL over its whole length



$$\text{Area of BM diagram} (A) = \frac{1}{3} \times L \times \left(\frac{wL^2}{2}\right) = \frac{wL^3}{6}$$

Therefore

$$\text{Maximum slope}(\theta) = \frac{A}{EI} = \frac{wL^3}{6EI} \qquad \text{(at free end)}$$

$$\text{Maximum deflection } (\delta) = \frac{A\bar{x}}{EI}$$

$$= \frac{\left(\frac{wL^3}{6}\right) \times \left(\frac{3}{4}L\right)}{EI} = \frac{wL^4}{8EI} \text{ (at free end)}$$

(iv) A simply supported beam with point load at mid-spam

Area of shaded BM diagram

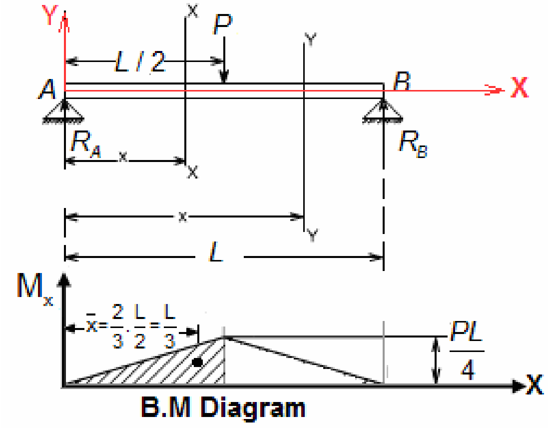
$$(A) = \frac{1}{2} \times \frac{L}{2} \times \frac{PL}{4} = \frac{PL^2}{16}$$

Therefore

$$\text{Maximum slope}(\theta) = \frac{A}{EI} = \frac{PL^2}{16EI} \quad \text{(at each ends)}$$

$$\text{Maximum deflection }(\delta) = \frac{A\bar{x}}{EI}$$

$$= \frac{\left(\frac{PL^2}{16} \times \frac{L}{3}\right)}{EI} = \frac{PL^3}{48EI} \quad \text{(at mid point)}$$



(v) A simply supported beam with UDL over its whole length

Area of BM diagram (shaded)

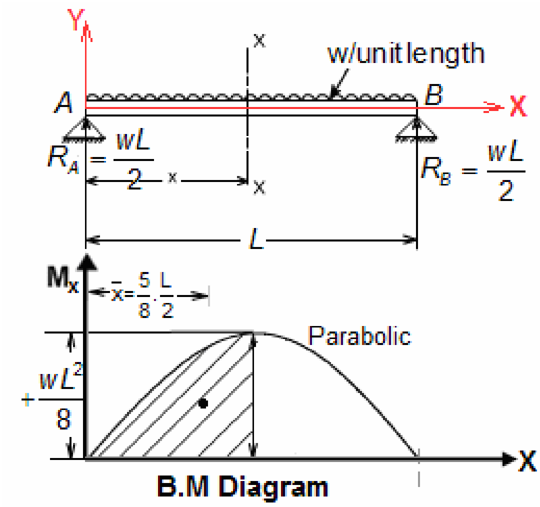
$$(A) = \frac{2}{3} \times \left(\frac{L}{2}\right) \times \left(\frac{wL^2}{8}\right) = \frac{wL^3}{24}$$

Therefore

$$\text{Maximum slope}(\theta) = \frac{A}{EI} = \frac{wL^3}{24EI} \quad \text{(at each ends)}$$

$$\text{Maximum deflection }(\delta) = \frac{A\bar{x}}{EI}$$

$$= \frac{\left(\frac{wL^3}{24}\right) \times \left(\frac{5}{8} \times \frac{L}{2}\right)}{EI} = \frac{5}{384} \frac{wL^4}{EI} \quad \text{(at mid point)}$$



9. Method of superposition

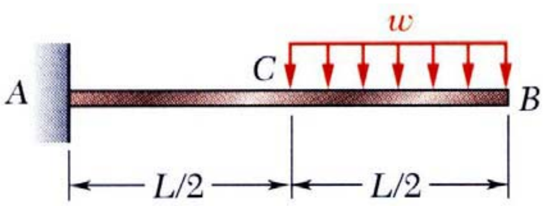
Assumptions:

- Structure should be linear
- Slope of elastic line should be very small.
- The deflection of the beam should be small such that the effect due to the shaft or rotation of the line of action of the load is neglected.

Principle of Superposition:

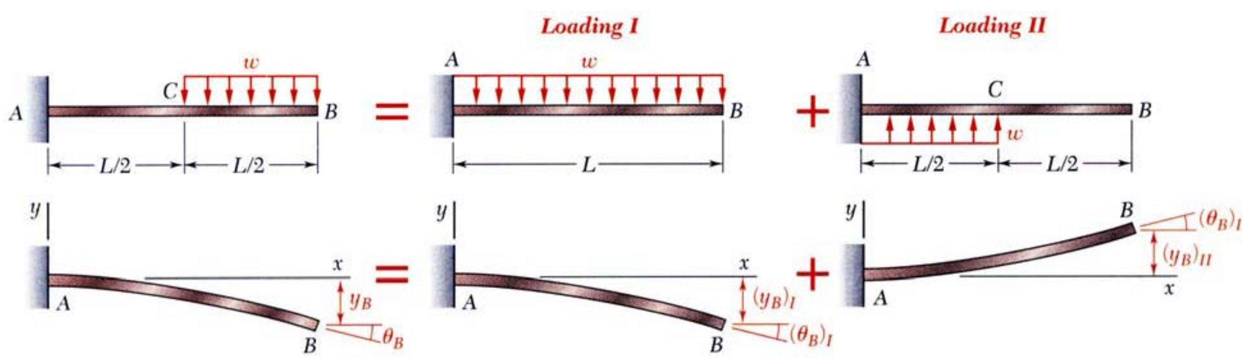
- Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

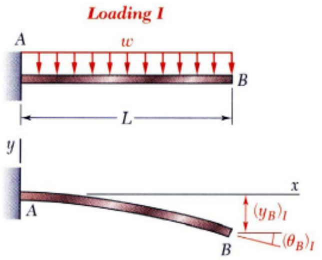
Example:



For the beam and loading shown, determine the slope and deflection at point *B*.

Superpose the deformations due to *Loading I* and *Loading II* as shown.



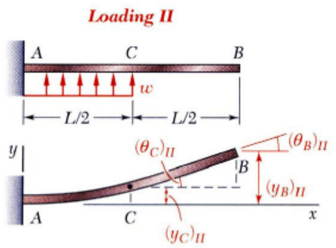


Loading I

$$(\theta_B)_I = -\frac{wL^3}{6EI} \qquad (y_B)_I = -\frac{wL^4}{8EI}$$

Loading II

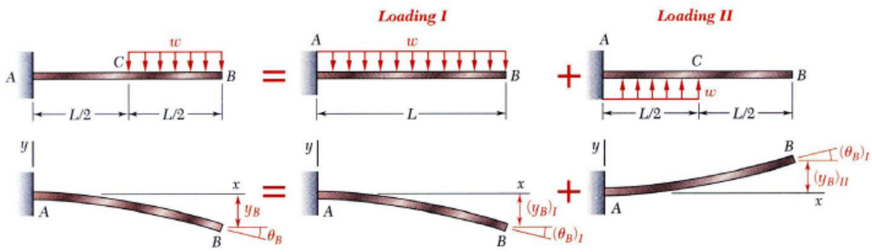
$$(\theta_C)_{II} = \frac{wL^3}{48EI} \qquad (y_C)_{II} = \frac{wL^4}{128EI}$$



In beam segment CB, the bending moment is zero and the elastic curve is a straight line.

$$(\theta_B)_{II} = (\theta_C)_{II} = \frac{wL^3}{48EI}$$

$$(y_B)_{II} = \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = \frac{7wL^4}{384EI}$$



Combine the two solutions,

$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI} \qquad \boxed{\theta_B = \frac{7wL^3}{48EI}}$$

$$y_B = (y_B)_I + (y_B)_{II} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI} \qquad \boxed{y_B = \frac{41wL^4}{384EI}}$$

10. Conjugate beam method

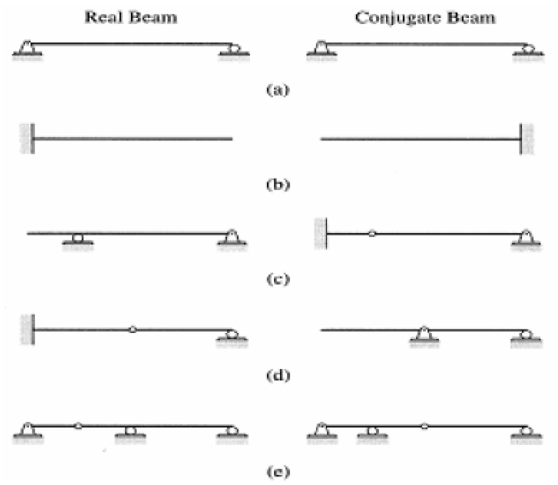
In the ***conjugate beam method***, the **length** of the conjugate beam is the same as the length of the actual beam, the **loading diagram** (showing the loads acting) on the conjugate beam is simply the bending-moment diagram of the actual beam divided by the flexural rigidity EI of the actual beam, and the **corresponding support condition** for the conjugate beam is given by the rules as shown below.

Corresponding support condition for the conjugate beam

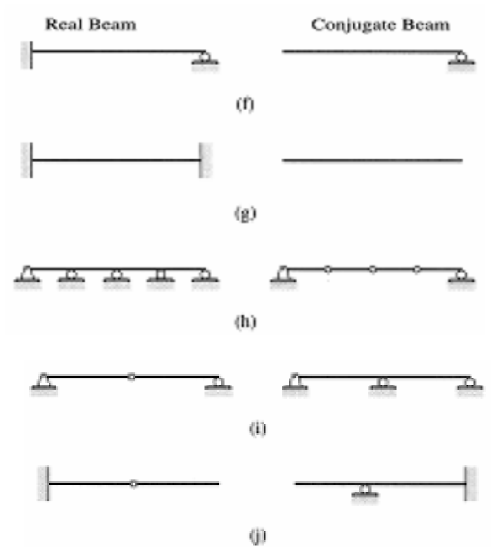
	Existing support condition of the **actual beam**	Corresponding support condition for the **conjugate beam**
Rule 1	Fixed end	Free end
Rule 2	Free end	Fixed end
Rule 3	Simple support at the end	Simple support at the end
Rule 4	Simple support *not* at the end	Unsupported hinge
Rule 5	Unsupported hinge	Simple support

Conjugates of Common Types of Real Beams

Conjugate beams for statically determinate real beams



Conjugate beams for Statically indeterminate real beams



By the conjugate beam method, the slope and deflection of the actual beam can be found by using the following two rules:

- The **slope** of the actual beam at any cross section is equal to the **shearing force** at the corresponding cross section of the conjugate beam.
- The **deflection** of the actual beam at any point is equal to the **bending moment** of the conjugate beam at the corresponding point.

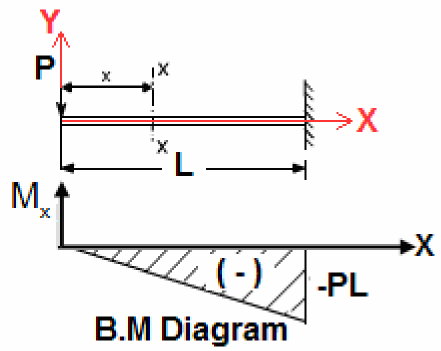
Procedure for Analysis

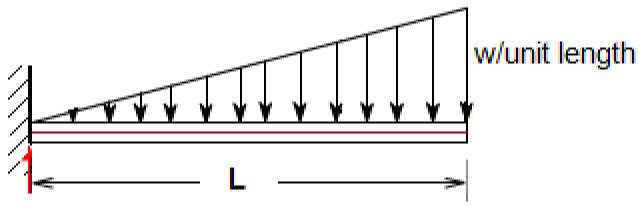
- Construct the **M / EI** diagram for the given (real) beam subjected to the specified (real) loading. If a combination of loading exists, you may use M-diagram by parts
- Determine the ***conjugate*** beam corresponding to the given real beam
- Apply the M / EI diagram as the **load on the conjugate** beam as per sign convention
- Calculate the **reactions** at the supports of the **conjugate** beam by applying equations of equilibrium and conditions
- Determine the ***shears*** in the ***conjugate*** beam at locations where **slopes** is desired in the **real** beam, $V_{conj} = \theta_{real}$
- Determine the ***bending moments*** in the ***conjugate*** beam at locations where **deflections** is desired in the **real** beam, $M_{conj} = y_{real}$

The method of double integration, method of superposition, moment-area theorems, and Castigliano's theorem are all well established methods for finding deflections of beams, but they require that the **boundary conditions** of the beams be known or specified. If not, all of them become *helpless*. However, the conjugate beam method is able to proceed and yield a solution for the possible deflections of the beam based on the **support conditions**, rather than the boundary conditions, of the beams.

(i) A Cantilever beam with a point load 'P' at its free end.

For Real Beam: At a section a distance 'x' from free end consider the forces to the left. Taking moments about the section gives (obviously to the left of the section) $M_x = -P.x$ (negative sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as negative according to the sign convention) so that the ***maximum bending moment*** occurs at the fixed end i.e. $M_{max} = -PL$ (at x = L)





Considering equilibrium we get, $M_A = \frac{wL^2}{3}$ and Reaction $(R_A) = \frac{wL}{2}$

Considering any cross-section XX which is at a distance of x from the fixed end.

At this point load $(W_x) = \frac{W}{L}.x$

Shear force $(\mathbf{V_x})$ $= R_A -$ area of triangle ANM

$$= \frac{wL}{2} - \frac{1}{2}.\left(\frac{w}{L}.x\right).x = +\frac{wL}{2} - \frac{wx^2}{2L}$$

$\therefore$ The shear force variation is parabolic.

at $x = 0$, $V_x = +\frac{wL}{2}$ i.e. Maximum shear force, $V_{max} = +\frac{wL}{2}$

at $x = L$, $V_x = 0$

Bending moment $(\mathbf{M_x})$ $= R_A.x - \frac{wx^2}{2L}.\frac{2x}{3} - M_A$

$$= \frac{wL}{2}.x - \frac{wx^3}{6L} - \frac{wL^2}{3}$$

$\therefore$ The bending moment variation is cubic

at $x = 0$, $M_x = -\frac{wL^2}{3}$ i.e.Maximum B.M. $\left(M_{max}\right) = -\frac{wL^2}{3}$.

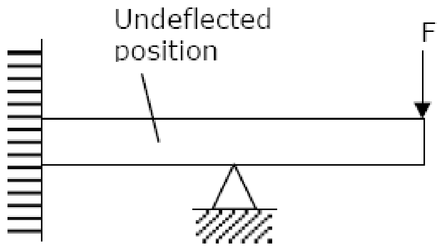
at $x = L$, $M_x = 0$

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Beam Deflection

GATE-1. A lean elastic beam of given flexural rigidity, EI, is loaded by a single force F as shown in figure. How many boundary conditions are necessary to determine the deflected centre line of the beam?



(a) 5 (b) 4

(c) 3 (d) 2

[GATE-1999]

GATE-1. Ans. (d) $EI\frac{d^2y}{dx^2} = M$. Since it is second order differential equation so we need two boundary conditions to solve it.

Double Integration Method

GATE-2. A simply supported beam carrying a concentrated load W at mid-span deflects by δ_1 under the load. If the same beam carries the load W such that it is distributed uniformly over entire length and undergoes a deflection δ_2 at the mid span. The ratio δ_1: δ_2 is: [IES-1995; GATE-1994]

(a) 2: 1 (b) $\sqrt{2}$: 1 (c) 1: 1 (d) 1: 2

GATE-2. Ans. (d) $\delta_1 = \frac{Wl^3}{48EI} =$ and $\delta_2 = \frac{5\left(\frac{W}{l}\right)l^4}{384EI} = \frac{5Wl^3}{384EI}$ Therefore δ_1: δ_2 = 5: 8

GATE-3. A simply supported laterally loaded beam was found to deflect more than a specified value. [GATE-2003]

Which of the following measures will reduce the deflection?

(a) Increase the area moment of inertia

(b) Increase the span of the beam

(c) Select a different material having lesser modulus of elasticity

(d) Magnitude of the load to be increased

GATE-3. Ans. (a) Maximum deflection (δ) = $\frac{Wl^3}{48EI}$

To reduce, δ, increase the area moment of Inertia.

Previous 20-Years IES Questions

Double Integration Method

IES-1. Consider the following statements: **[IES-2003]**
In a cantilever subjected to a concentrated load at the free end
1. **The bending stress is maximum at the free end**
2. **The maximum shear stress is constant along the length of the beam**
3. **The slope of the elastic curve is zero at the fixed end**

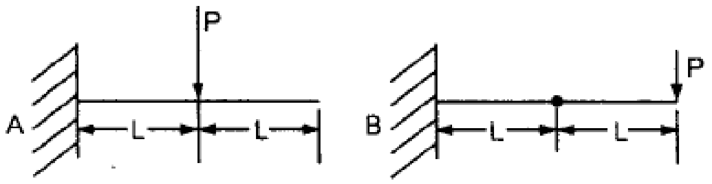
Which of these statements are correct?
(a) 1, 2 and 3 (b) 2 and 3 (c) 1 and 3 (d) 1 and 2

IES-1. Ans. (b)

IES-2. A cantilever of length L, moment of inertia I. Young's modulus E carries a concentrated load W at the middle of its length. The slope of cantilever at the free end is: **[IES-2001]**
(a) $\dfrac{WL^2}{2EI}$ (b) $\dfrac{WL^2}{4EI}$ (c) $\dfrac{WL^2}{8EI}$ (d) $\dfrac{WL^2}{16EI}$

IES-2. Ans. (c) $\theta = \dfrac{W\left(\dfrac{L}{2}\right)^2}{2EI} = \dfrac{WL^2}{8EI}$

IES-3. The two cantilevers A and B shown in the figure have the same uniform cross-section and the same material. Free end deflection of cantilever 'A' is δ.



[IES-2000]

The value of mid- span deflection of the cantilever 'B' is:
(a) $\dfrac{1}{2}\delta$ (b) $\dfrac{2}{3}\delta$ (c) δ (d) 2δ

IES-3. Ans. (c) $\delta = \dfrac{WL^3}{3EI} + \left(\dfrac{WL^2}{2EI}\right)L = \dfrac{5WL^3}{6EI}$

$$y_{mid} = \frac{W}{EI}\left(\frac{2Lx^2}{2} - \frac{x^3}{6}\right)_{at\,x=L} = \frac{5WL^3}{6EI} = \delta$$

IES-4. A cantilever beam of rectangular cross-section is subjected to a load W at its free end. If the depth of the beam is doubled and the load is halved, the deflection of the free end as compared to original deflection will be: **[IES-1999]**
(a) Half (b) One-eighth (c) One-sixteenth (d) Double

IES-4. Ans. (c) Deflectionin cantilever $= \dfrac{Wl^3}{3EI} = \dfrac{Wl^3 \times 12}{3Eah^3} = \dfrac{4Wl^3}{Eah^3}$

If h is doubled, and W is halved, New deflection $= \dfrac{4Wl^3}{2Ea(2h)^3} = \dfrac{1}{16} \times \dfrac{4Wl^3}{Eah^3}$

IES-5. A simply supported beam of constant flexural rigidity and length 2L carries a concentrated load 'P' at its mid-span and the deflection under the load is δ. If a cantilever beam of the same flexural rigidity and length 'L' is subjected to load 'P' at its free end, then the deflection at the free end will be: **[IES-1998]**
(a) $\dfrac{1}{2}\delta$ (b) δ (c) 2δ (d) 4δ

IES-5. Ans. (c) δ for simply supported beam $= \dfrac{W(2L)^3}{48EI} = \dfrac{WL^3}{6EI}$

and deflection for Cantilever $= \dfrac{WL^3}{3EI} = 2\delta$

IES-6. **Two identical cantilevers are loaded as shown in the respective figures. If slope at the free end of the cantilever in figure E is θ, the slope at free and of the cantilever in figure F will be:**

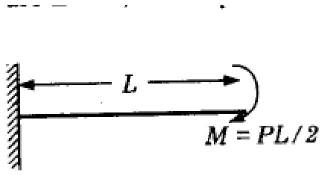


Figure E

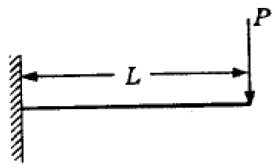


Figure F

[IES-1997]

(a) $\frac{1}{3}\theta$ (b) $\frac{1}{2}\theta$ (c) $\frac{2}{3}\theta$ (d) θ

IES-6. Ans. (d) When a B. M is applied at the free end of cantilever, $\theta = \dfrac{ML}{EI} = \dfrac{(PL/2)L}{EI} = \dfrac{PL^2}{2EI}$

When a cantilever is subjected to a single concentrated load at free end, then $\theta = \dfrac{PL^2}{2EI}$

IES-7. **A cantilever beam carries a load W uniformly distributed over its entire length. If the same load is placed at the free end of the same cantilever, then the ratio of maximum deflection in the first case to that in the second case will be:** **[IES-1996]**

(a) 3/8 (b) 8/3 (c) 5/8 (d) 8/5

IES-7. Ans. (a) $\dfrac{Wl^3}{8EI} \div \dfrac{Wl^3}{3EI} = \dfrac{3}{8}$

IES-8. **The given figure shows a cantilever of span 'L' subjected to a concentrated load 'P' and a moment 'M' at the free end. Deflection at the free end is given by**

[IES-1996]

(a) $\dfrac{PL^2}{2EI} + \dfrac{ML^2}{3EI}$ (b) $\dfrac{ML^2}{2EI} + \dfrac{PL^3}{3EI}$ (c) $\dfrac{ML^2}{3EI} + \dfrac{PL^3}{2EI}$ (d) $\dfrac{ML^2}{2EI} + \dfrac{PL^3}{48EI}$

IES-8. Ans. (b)

IES-9. **For a cantilever beam of length 'L', flexural rigidity EI and loaded at its free end by a concentrated load W, match List I with List II and select the correct answer.** **[IES-1996]**

List I	**List II**
A. Maximum bending moment	**1.** Wl
B. Strain energy	**2.** $Wl^2/2EI$
C. Maximum slope	**3.** $Wl^3/3EI$
D. Maximum deflection	**4.** $W^2l^2/6EI$

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	1	4	3	2	(b)	1	4	2	3
(c)	4	2	1	3	(d)	4	3	1	2

IES-9. Ans. (b)

IES-10. **Maximum deflection of a cantilever beam of length 'l' carrying uniformly distributed load w per unit length will be:** **[IES- 2008]**

(a) $wl^4/(EI)$ (b) $w\,l^4/(4\,EI)$ (c) $w\,l^4/(8\,EI)$ (d) $w\,l^4/(384\,EI)$

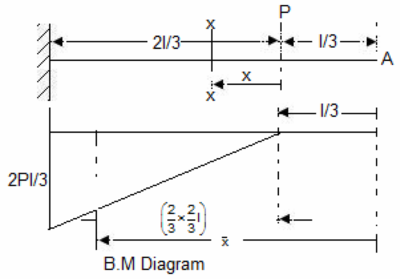
[Where E = modulus of elasticity of beam material and I = moment of inertia of beam cross-section]

IES-10. Ans. (c)

IES-11. A cantilever beam of length 'l' is subjected to a concentrated load P at a distance of l/3 from the free end. What is the deflection of the free end of the beam? (EI is the flexural rigidity) [IES-2004]

(a) $\frac{2Pl^3}{81EI}$ (b) $\frac{3Pl^3}{81EI}$ (c) $\frac{14Pl^3}{81EI}$ (d) $\frac{15Pl^3}{81EI}$

IES-11. Ans. (d) Moment Area method gives us



$$\delta_A = \frac{\text{Area}}{EI}\bar{x} = \frac{\frac{1}{2}\times\left(\frac{2Pl}{3}\right)\times\left(\frac{2l}{3}\right)\times\left(\frac{l}{3}+\frac{4}{9}l\right)}{EI}$$

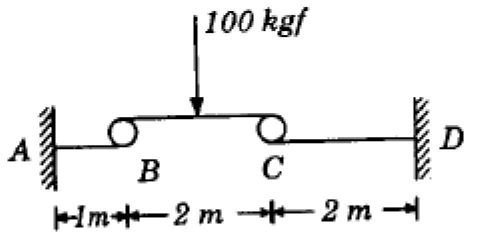
$$= \frac{Pl^3}{EI}\times\frac{2}{9}\times\frac{7}{9} = \frac{14}{81}\frac{Pl^3}{EI}$$

$$\text{Alternatively } Y_{max} = \frac{Wa^2}{EI}\left\{\frac{l}{2}-\frac{a}{6}\right\} = \frac{W\left(\frac{2l}{3}\right)^2}{EI}\left\{\frac{l}{2}-\frac{2l/3}{6}\right\}$$

$$= \frac{Wl^3}{EI}\times\frac{4}{9}\times\frac{(9-2)}{18}$$

$$= \frac{14}{81}\frac{Wl^3}{EI}$$

IES-12. A 2 m long beam BC carries a single concentrated load at its mid-span and is simply supported at its ends by two cantilevers AB = 1 m long and CD = 2 m long as shown in the figure. The shear force at end A of the cantilever AB will be



(a) Zero (b) 40 kg
(c) 50 kg (d) 60 kg

[IES-1997]

IES-12. Ans. (c) Reaction force on B and C is same 100/2 = 50 kg. And we know that shear force is same throughout its length and equal to load at free end.

IES-13. Assertion (A): In a simply supported beam subjected to a concentrated load P at mid-span, the elastic curve slope becomes zero under the load. [IES-2003]
Reason (R): The deflection of the beam is maximum at mid-span.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-13. Ans. (a)

IES-14. At a certain section at a distance 'x' from one of the supports of a simply supported beam, the intensity of loading, bending moment and shear force arc W_x, M_x and V_x respectively. If the intensity of loading is varying continuously along the length of the beam, then the *invalid* relation is: [IES-2000]

$$(a)\,\text{Slope}\,Q_x = \frac{M_x}{V_x} \quad (b)\,V_x = \frac{dM_x}{dx} \quad (c)\,W_x = \frac{d^2M_x}{dx^2} \quad (d)\,W_x = \frac{dV_x}{dx}$$

IES-14. Ans. (a)

IES-15. The bending moment equation, as a function of distance x measured from the left end, for a simply supported beam of span L m carrying a uniformly distributed load of intensity w N/m will be given by [IES-1999]

$(a)\ M=\frac{wL}{2}(L-x)-\frac{w}{2}(L-x)^3\ \text{Nm}$ $(b)\ M=\frac{wL}{2}(x)-\frac{w}{2}(x)^2\ \text{Nm}$

$(c)\ M=\frac{wL}{2}(L-x)^2-\frac{w}{2}(L-x)^3\ \text{Nm}$ $(d)\ M=\frac{wL}{2}(x)^2-\frac{wLx}{2}\ \text{Nm}$

IES-15. Ans. (b)

IES-16. A simply supported beam with width 'b' and depth 'd' carries a central load W and undergoes deflection δ at the centre. If the width and depth are interchanged, the deflection at the centre of the beam would attain the value [IES-1997]

$(a)\ \frac{d}{b}\delta$ $(b)\ \left(\frac{d}{b}\right)^2\delta$ $(c)\ \left(\frac{d}{b}\right)^3\delta$ $(d)\ \left(\frac{d}{b}\right)^{3/2}\delta$

IES-16. Ans. (b) Deflection at center $\delta=\frac{Wl^3}{48EI}=\frac{Wl^3}{48E\left(\frac{bd^3}{12}\right)}$

$$\text{In second case, deflection}=\delta'=\frac{Wl^3}{48EI'}=\frac{Wl^3}{48E\left(\frac{db^3}{12}\right)}=\frac{Wl^3}{48E\left(\frac{bd^3}{12}\right)}\frac{d^2}{b^2}=\frac{d^2}{b^2}\delta$$

IES-17. A simply supported beam of rectangular section 4 cm by 6 cm carries a mid-span concentrated load such that the 6 cm side lies parallel to line of action of loading; deflection under the load is δ. If the beam is now supported with the 4 cm side parallel to line of action of loading, the deflection under the load will be: [IES-1993]

(a) 0.44 δ (b) 0.67 δ (c) 1.5 δ (d) 2.25 δ

IES-17. Ans. (d) Use above explanation

IES-18. A simply supported beam carrying a concentrated load W at mid-span deflects by δ_1 under the load. If the same beam carries the load W such that it is distributed uniformly over entire length and undergoes a deflection δ_2 at the mid span. The ratio δ_1: δ_2 is: [IES-1995; GATE-1994]

(a) 2: 1 (b) $\sqrt{2}$: 1 (c) 1: 1 (d) 1: 2

IES-18. Ans. (d) $\delta_1=\frac{Wl^3}{48EI}=$ and $\delta_2=\frac{5\left(\frac{W}{l}\right)l^4}{384EI}=\frac{5Wl^3}{384EI}$ Therefore δ_1: δ_2 = 5: 8

Moment Area Method

IES-19. Match List-I with List-II and select the correct answer using the codes given below the Lists: [IES-1997]

List-I	List-II
A. Toughness	**1.** Moment area method
B. Endurance strength	**2.** Hardness
C. Resistance to abrasion	**3.** Energy absorbed before fracture in a tension test
D. Deflection in a beam	**4.** Fatigue loading

Code:	A	B	C	D		A	B	C	D
(a)	4	3	1	2	(b)	4	3	2	1
(c)	3	4	2	1	(d)	3	4	1	2

IES-19. Ans. (c)

Previous 20-Years IAS Questions

Slope and Deflection at a Section

IAS-1. Which one of the following is represented by the area of the S.F diagram from one end upto a given location on the beam? [IAS-2004]

(a) B.M. at the location (b) Load at the location
(c) Slope at the location (d) Deflection at the location

IAS-1. Ans. (a)

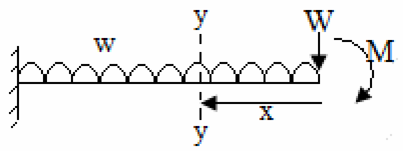
Double Integration Method

IAS-2. Which one of the following is the correct statement? [IAS-2007]

If for a beam $\frac{dM}{dx} = 0$ for its whole length, the beam is a cantilever:

(a) Free from any load (b) Subjected to a concentrated load at its free end
(c) Subjected to an end moment (d) Subjected to a udl over its whole span

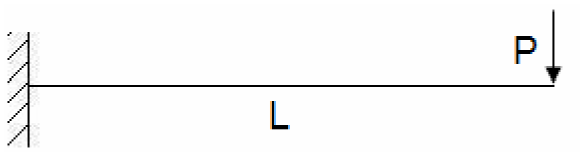
IAS-2. Ans. (c) udl or point load both vary with x. But if we apply Bending Moment (M) = const. and $\frac{dM}{dx} = 0$



IAS-3. In a cantilever beam, if the length is doubled while keeping the cross-section and the concentrated load acting at the free end the same, the deflection at the free end will increase by [IAS-1996]

(a) 2.66 times (b) 3 times (c) 6 times (d) 8 times

IAS-3. Ans. (d)



$$\delta = \frac{PL^3}{3EI} \qquad \therefore \delta \propto L^3 \qquad \therefore \frac{\delta_2}{\delta_1} = \left(\frac{L_2}{L_1}\right)^3 = 8$$

Conjugate Beam Method

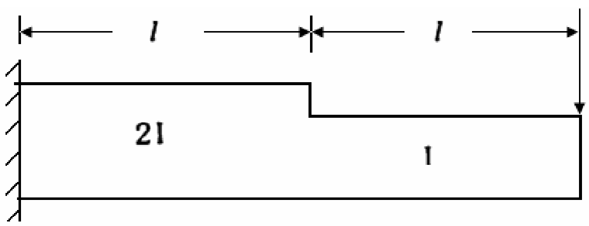
IAS-4. By conjugate beam method, the slope at any section of an actual beam is equal to: [IAS-2002]

(a) EI times the S.F. of the conjugate beam (b) EI times the B.M. of the conjugate beam
(c) S.F. of conjugate beam (d) B.M. of the conjugate beam

IAS-4. Ans. (c)

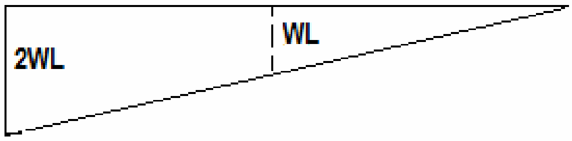
IAS-5. I = 375 × 10^{-6} m^4; l = 0.5 m
E = 200 GPa
Determine the stiffness of the beam shown in the above figure

(a) 12 × 10^{10} N/m
(b) 10 × 10^{10} N/m
(c) 4 × 10^{10} N/m
(d) 8 × 10^{10} N/m

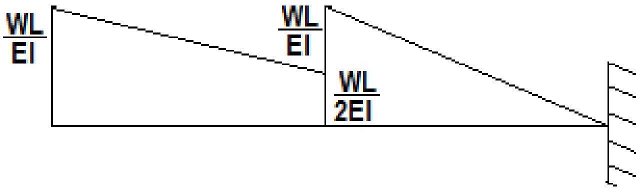


[IES-2002]

IAS-5. Ans. (c) Stiffness means required load for unit deformation. BMD of the given beam



Loading diagram of conjugate beam



The deflection at the free end of the actual beam = BM of the at fixed point of conjugate beam

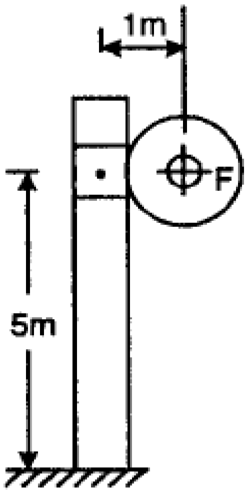
$$y = \left(\frac{1}{2}\times L\times\frac{ML}{EI}\right)\times\frac{2L}{3}+\left(\frac{WL}{2EI}\times L\right)\times\left(L+\frac{L}{2}\right)+\left(\frac{1}{2}\times L\times\frac{WL}{2EI}\right)\times\left(L+\frac{2L}{3}\right)=\frac{3WL^3}{2EI}$$

Or stiffness = $\dfrac{W}{y}=\dfrac{2EI}{3L^3}=\dfrac{2\times(200\times10^9)\times(375\times10^{-6})}{3\times(0.5)^3}=4\times10^{10}\ N/m$

Previous Conventional Questions with Answers

Conventional Question GATE-1999

***Question*:** **Consider the signboard mounting shown in figure below. The wind load acting perpendicular to the plane of the figure is F = 100 N. We wish to limit the deflection, due to bending, at point A of the hollow cylindrical pole of outer diameter 150 mm to 5 mm. Find the wall thickness for the pole. [Assume E = 2.0 X 10^{11} N/m^2]**



***Answer*:** Given: F = 100 N; d_0 = 150 mm, 0.15 my = 5 mm; E = 2.0 X $1O^{11}$ N/m^2

Thickness of pole, t

The system of signboard mounting can be considered as a cantilever loaded at A i.e. W = 100 N and also having anticlockwise moment of M = 100 x 1 = 100 Nm at the free end. Deflection of cantilever having concentrated load at the free end,

$$y = \frac{WL^3}{3EI} + \frac{ML^2}{2EI}$$

$$5 \times 10^{-3} = \frac{100 \times 5^3}{3 \times 2.0 \times 10^{11} \times I} + \frac{100 \times 5^3}{2 \times 2.0 \times 10^{11} \times I}$$

or $$I = \frac{1}{5 \times 10^{-3}} \left[\frac{100 \times 5^3}{3 \times 2.0 \times 10^{11}} + \frac{100 \times 5^3}{2 \times 2.0 \times 10^{11}} \right] = 5.417 \times 10^{-6}\ m^4$$

But $$I = \frac{\pi}{64}\left(d_0^4 - d_i^4\right)$$

$\therefore$ $$5.417 \times 10^{-6} = \frac{\pi}{64}\left(0.15^4 - d_i^4\right)$$

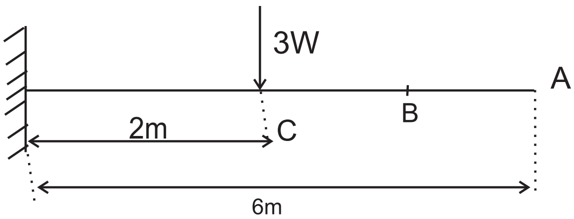
or $$d_i = 0.141\text{m or } 141\ \text{mm}$$

$\therefore$ $$t = \frac{d_0 - d_i}{2} = \frac{150 - 141}{2} = 4.5\ \text{mm}$$

Conventional Question IES-2003

***Question*:** **Find the slope and deflection at the free end of a cantilever beam of length 6m as loaded shown in figure below, using method of superposition. Evaluate their numerical value using E = 200 GPa, I = 1×10^{-4} m^4 and W = 1 kN.**

Answer: We have to use superposition theory.



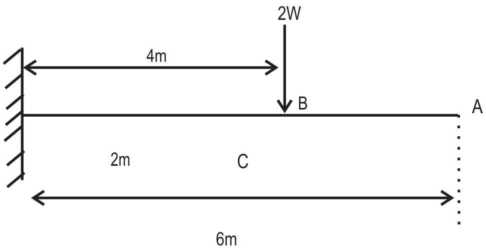
1st consider

$$\delta_c = \frac{PL^3}{3EI} = \frac{(3W)\times 2^3}{3EI} = \frac{8W}{EI}$$

$$\theta_c = \frac{PL^2}{2EI} = \frac{(3W).2^2}{2EI} = \frac{6W}{EI}$$

Deflection at A due to this load(δ_1) = $\delta_c + \theta_c.(6-2) = \frac{8W}{EI} + \frac{6W}{EI}\times 4 = \frac{32W}{EI}$

2nd consider:



$$\delta_B = \frac{(2W)\times 4^3}{3EI} = \frac{128W}{3EI}$$

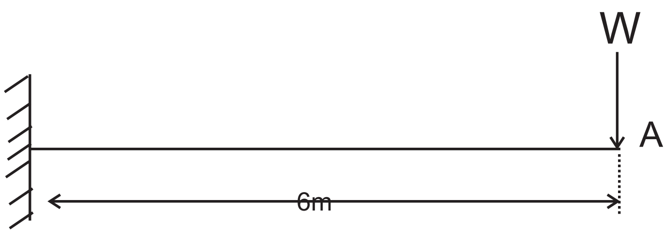
$$\theta_B = \frac{(2W)\times 4^2}{2EI} = \frac{16W}{EI}$$

Deflection at A due to this load(δ_2)

$$=\delta_B + \theta_B \times (6-4) = \frac{224W}{3EI}$$

.

3rd consider :



$$(\delta_3) = \delta_A = \frac{W\times 6^3}{3EI} = \frac{72W}{EI}$$

$$\theta_A = \frac{W\times 6^2}{2EI} = \frac{18W}{EI}$$

Apply superpositioning formula

$$\theta = \theta_A + \theta_B + \theta_c = \frac{6W}{EI} + \frac{16W}{EI} + \frac{18W}{EI} = \frac{40W}{EI} = \frac{40\times(10^3)}{(200\times 10^9)\times 10^{-4}}$$

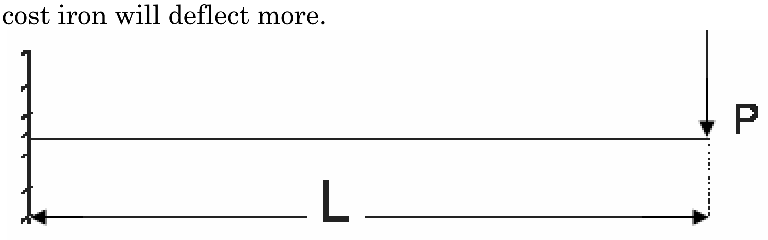
$$\delta = \delta_1 + \delta_2 + \delta_3 = \frac{32W}{EI} + \frac{224W}{3EI} + \frac{72W}{EI} = \frac{40W}{EI} = \frac{563\times W}{3EI}$$

$$= \frac{563\times(10^3)}{3\times(200\times 10^9)\times(10^{-4})} = 8.93\,\text{mm}$$

Conventional Question IES-2002

Question: **If two cantilever beams of identical dimensions but made of mild steel and grey cast iron are subjected to same point load at the free end, within elastic limit, which one will deflect more and why?**

Answer: Grey cost iron will deflect more.



We know that a cantilever beam of length 'L' end load 'P' will deflect at free end

$$(\delta) = \frac{PL^3}{3EI}$$

$$\therefore \delta \propto \frac{1}{E}$$

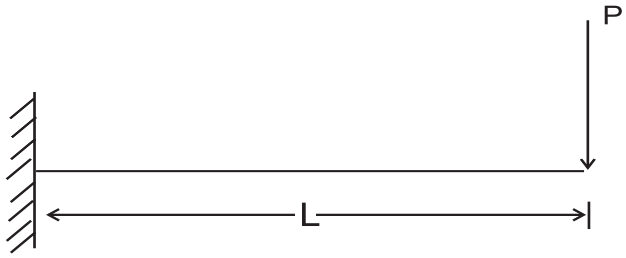
$E_{Cast\ Iron} \simeq 125\,GPa$ and $E_{Mild\ steel} \simeq 200\ GPa$

Conventional Question IES-1997

***Question*:** **A uniform cantilever beam (EI = constant) of length L is carrying a concentrated load P at its free end. What would be its slope at the (i) Free end and (ii) Built in end**

***Answer*:** (i) Free end, $\theta = \dfrac{PL^2}{2EI}$

(ii) Built-in end, $\theta = 0$



6. Bending Stress in Beam

Theory at a Glance (for IES, GATE, PSU)

6.1 Euler Bernoulli's Equation or (Bending stress formula) or Bending Equation

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

Where σ = Bending Stress

M = Bending Moment

I = Moment of Inertia

E = Modulus of elasticity

R = Radius of curvature

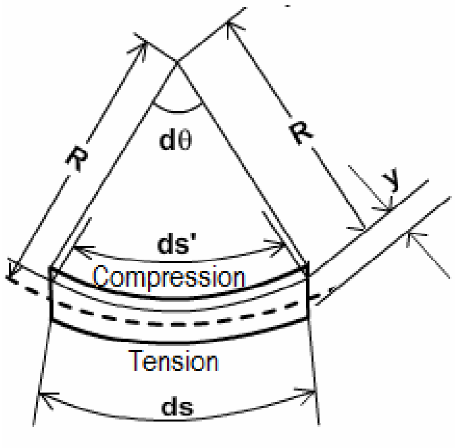
y = Distance of the fibre from NA (Neutral axis)

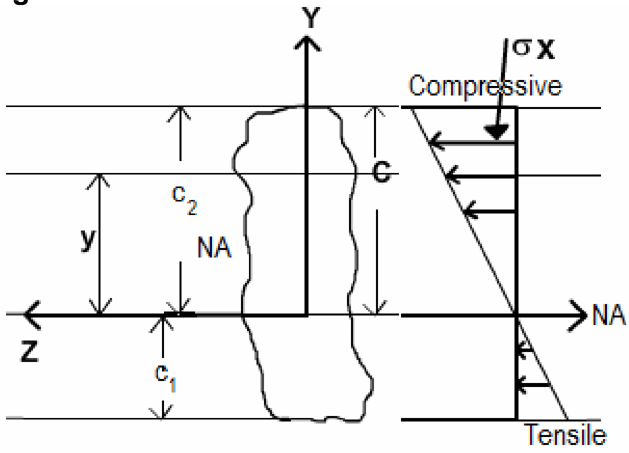
6.2 Assumptions in Simple Bending Theory

All of the foregoing theory has been developed for the case of pure bending i.e. constant B.M along the length of the beam. In such case

- The shear force at each c/s is zero.
- Normal stress due to bending is only produced.
- Beams are initially straight
- The material is homogenous and isotropic i.e. it has a uniform composition and its mechanical properties are the same in all directions
- The stress-strain relationship is linear and elastic
- Young's Modulus is the same in tension as in compression
- Sections are symmetrical about the plane of bending
- Sections which are plane before bending remain plane after bending

6.3





$$\sigma_{\max} = \sigma_t = \frac{Mc_1}{I}$$

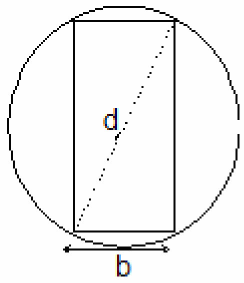
$$\sigma_{\min} = \sigma_c = \frac{Mc_2}{I}$$ (Minimum in sense of sign)

6.4 Section Modulus (Z)

$$Z = \frac{I}{y}$$

- Z is a function of beam c/s only
- Z is other name of the strength of the beam
- The strength of the beam sections depends mainly on the section modulus
- The flexural formula may be written as, $\sigma = \frac{M}{Z}$
- Rectangular c/s of width is "b" & depth "h" with sides horizontal, Z = $\frac{bh^2}{6}$
- Square beam with sides horizontal, Z = $\frac{a^3}{6}$
- Square c/s with diagonal horizontal, Z = $\frac{a^3}{6\sqrt{2}}$
- Circular c/s of diameter "d", Z = $\frac{\pi d^3}{32}$

A log diameter "d" is available. It is proposed to cut out a strongest beam from it. Then



$$Z = \frac{b(d^2 - b^2)}{6}$$

Therefore, $Z_{max} = \frac{bd^3}{9}$ for $b = \frac{d}{\sqrt{3}}$

6.5 Flexural Rigidity (EI)

Reflects both

- Stiffness of the material (measured by E)
- Proportions of the c/s area (measured by I)

6.6 Axial Rigidity = EA

6.7 Beam of uniform strength

It is one is which the maximum bending stress is same in every section along the longitudinal axis.

For it $M \alpha \text{ bh}^2$

Where b = Width of beam

h = Height of beam

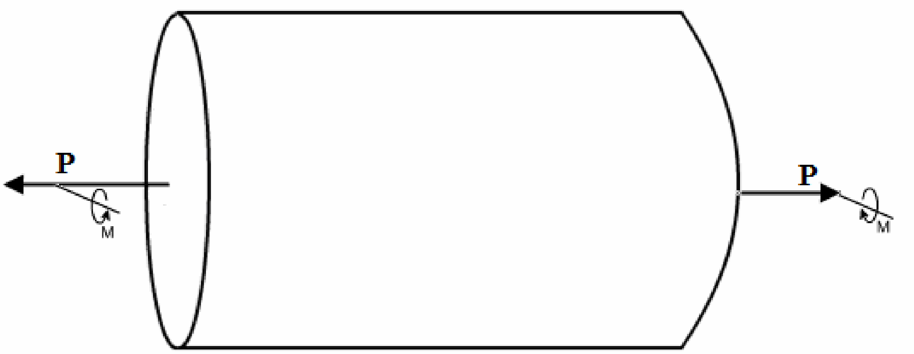
To make Beam of uniform strength the section of the beam may be varied by

- Keeping the width constant throughout the length and varying the depth, *(Most widely used)*
- Keeping the depth constant throughout the length and varying the width
- By varying both width and depth suitably.

6.8 Bending stress due to additional Axial thrust (P).

A shaft may be subjected to a combined bending and axial thrust. This type of situation arises in various machine elements.

If P = Axial thrust



Then direct stress (σ_d) = P / A (stress due to axial thrust)

This direct stress (σ_d) may be tensile or compressive depending upon the load P is tensile or compressive.

And the bending stress (σ_b) = $\dfrac{My}{I}$ is varying linearly from zero at centre and extremum (minimum or maximum) at top and bottom fibres.

If P is compressive then

- At top fibre $\sigma = \dfrac{P}{A} + \dfrac{My}{I}$ (compressive)
- At mid fibre $\sigma = \dfrac{P}{A}$ (compressive)
- At bottom fibre $\sigma = \dfrac{P}{A} - \dfrac{My}{I}$ (compressive)

6.9 Load acting eccentrically to one axis

- $\sigma_{\max} = \dfrac{P}{A} + \dfrac{(P \times e)y}{I}$ where 'e' is the eccentricity at which 'P' is act.
- $\sigma_{\min} = \dfrac{P}{A} - \dfrac{(P \times e)y}{I}$

Condition for No tension in any section

- For no tension in any section, the eccentricity must not exceed $\dfrac{2k^2}{d}$

[Where d = depth of the section; k = radius of gyration of c/s]

- For rectangular section (b x h) , $e \le \dfrac{h}{6}$ i.e load will be $2e = \dfrac{h}{3}$ of the middle section.
- For circular section of diameter 'd' , $e \le \dfrac{d}{8}$ i.e. diameter of the kernel, $2e = \dfrac{d}{4}$

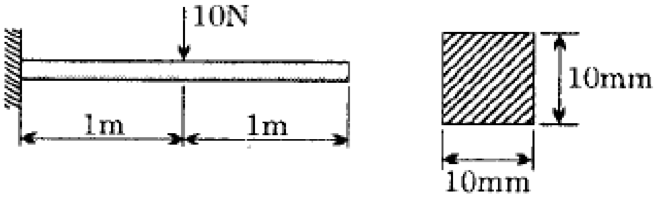
For hollow circular section of diameter 'd' , $e \le \dfrac{D^2 + d^2}{8D}$ i.e. diameter of the kernel, $2e \le \dfrac{D^2 + d^2}{4D}$.

OBJECTIVE QUESTIONS (GATE, IES, IAS)

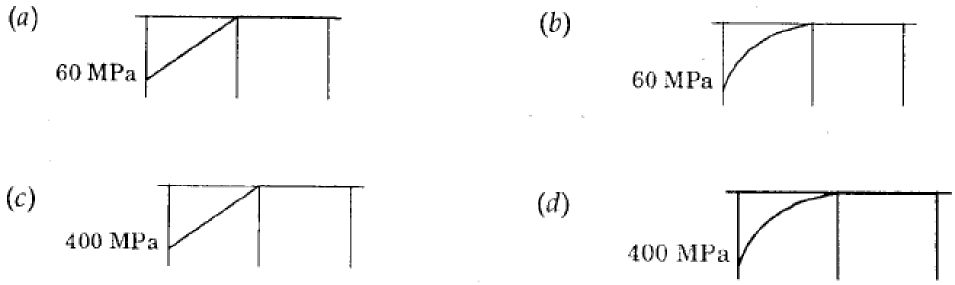
Previous 20-Years GATE Questions

Bending equation

GATE-1. A cantilever beam has the square cross section 10mm × 10 mm. It carries a transverse load of 10 N. Considering only the bottom fibres of the beam, the correct representation of the longitudinal variation of the bending stress is:



[GATE-2005]



GATE-1. Ans. (a) $M_x = P.x \qquad \dfrac{M}{I} = \dfrac{\sigma}{y} \quad \text{or } \sigma = \dfrac{My}{I} = \dfrac{10 \times (x) \times 0.005}{\dfrac{(0.01)^4}{12}} = 60.(x)\text{ MPa}$

$\text{At } x = 0; \quad \sigma = 0$

$\text{At } x = 1\text{m}; \quad \sigma = 60\text{MPa}$

And it is linear as $\sigma \propto x$

GATE-2. Two beams, one having square cross section and another circular cross-section, are subjected to the same amount of bending moment. If the cross sectional area as well as the material of both the beams are the same then [GATE-2003]

(a) Maximum bending stress developed in both the beams is the same
(b) The circular beam experiences more bending stress than the square one
(c) The square beam experiences more bending stress than the circular one
(d) As the material is same both the beams will experience same deformation

GATE-2. Ans. (b) $\dfrac{M}{I} = \dfrac{E}{\rho} = \dfrac{\sigma}{y}; \qquad \text{or } \sigma = \dfrac{My}{I};$

$$\sigma_{sq} = \frac{M\left(\dfrac{a}{2}\right)}{\dfrac{1}{12}a.a^3} = \frac{6M}{a^3}; \qquad \sigma_{cir} = \frac{M\left(\dfrac{d}{2}\right)}{\dfrac{\pi d^4}{64}} = \frac{32M}{\pi d^3} = \frac{4\pi\sqrt{\pi}M}{a^3} = \frac{22.27M}{a^3} \quad \left[\because \frac{\pi d^2}{4} = a^2\right]$$

$\therefore\ \sigma_{sq} < \sigma_{cir}$

Section Modulus

GATE-3. Match the items in Columns I and II. [GATE-2006]

Column-I	**Column-II**
P. Addendum	**1.** Cam
Q. Instantaneous centre of velocity	**2.** Beam

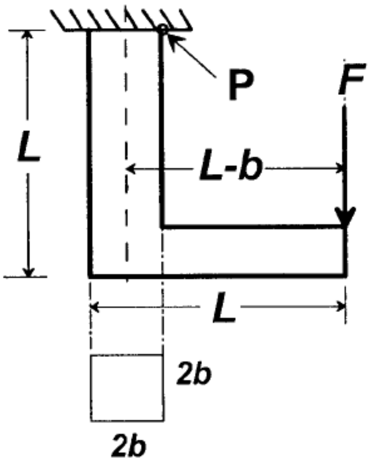
R. Section modulus — **3.** Linkage
S. Prime circle — **4.** Gear

(a) P – 4, Q – 2, R – 3, S – 1
(b) P – 4, Q – 3, R – 2, S – 1
(c) P – 3, Q – 2, R – 1, S – 4
(d) P – 3, Q – 4, R – 1, S – 2

GATE-3. Ans. (b)

Combined direct and bending stress

GATE-4. For the component loaded with a force *F* as shown in the figure, the axial stress at the corner point P is: [GATE-2008]



(a) $\frac{F(3L-b)}{4b^3}$ (b) $\frac{F(3L+b)}{4b^3}$ (c) $\frac{F(3L-4b)}{4b^3}$ (d) $\frac{F(3L-2b)}{4b^3}$

GATE-4. Ans. (d) Total Stress = Direct stress + Stress due to Moment

$$= \frac{P}{A} + \frac{My}{I} = \frac{F}{4b^2} + \frac{F(L-b)\times b}{\frac{2b\times(b)^3}{12}}$$

Previous 20-Years IES Questions

Bending equation

IES-1. Beam A is simply supported at its ends and carries udl of intensity w over its entire length. It is made of steel having Young's modulus E. Beam B is cantilever and carries a udl of intensity w/4 over its entire length. It is made of brass having Young's modulus E/2. The two beams are of same length and have same cross-sectional area. If σ_A and σ_B denote the maximum bending stresses developed in beams A and B, respectively, then which one of the following is correct? [IES-2005]

(a) σ_A/σ_B
(b) $\sigma_A/\sigma_B < 1.0$
(c) $\sigma_A/\sigma_B > 1.0$
(d) σ_A/σ_B depends on the shape of cross-section

IES-1. Ans. (d) Bending stress $(\sigma) = \frac{My}{I}$, y and I both depends on the

Shape of cross – sec tion so $\frac{\sigma_A}{\sigma_B}$ depends on the shape of cross – sec tion

IES-2. If the area of cross-section of a circular section beam is made four times, keeping the loads, length, support conditions and material of the beam unchanged, then the qualities (List-I) will change through different factors (List-II). Match the List-I with the List-II and select the correct answer using the code given below the Lists: [IES-2005]

List-I	**List-II**
A. Maximum BM	**1.** 8

B. Deflection					**2.** 1				
C. Bending Stress					**3.** 1/8				
D. Section Modulus					**4.** 1/16				
Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	3	1	2	4	(b)	2	4	3	1
(c)	3	4	2	1	(d)	2	1	3	4

IES-2. Ans. (b) Diameter will be double, D = 2d.

A. Maximum BM will be unaffected

B. deflection ratio $\dfrac{EI_1}{EI_2} = \left(\dfrac{d}{4}\right)^4 = \dfrac{1}{16}$

C. Bending stress $\sigma = \dfrac{My}{I} = \dfrac{M(d/2)}{\dfrac{\pi d^4}{64}}$ or Bending stress ratio $= \dfrac{\sigma_2}{\sigma_1} = \left(\dfrac{d}{D}\right)^3 = \dfrac{1}{8}$

D. Selection Modulus ratio $= \dfrac{Z_2}{Z_1} = \dfrac{I_2}{y_1} \times \dfrac{y_1}{I_1} = \left(\dfrac{D}{d}\right)^3 = 8$

IES-3. Consider the following statements in case of beams: [IES-2002]
1. **Rate of change of shear force is equal to the rate of loading at a particular section**
2. **Rate of change of bending moment is equal to the shear force at a particular suction.**
3. **Maximum shear force in a beam occurs at a point where bending moment is either zero or bending moment changes sign**

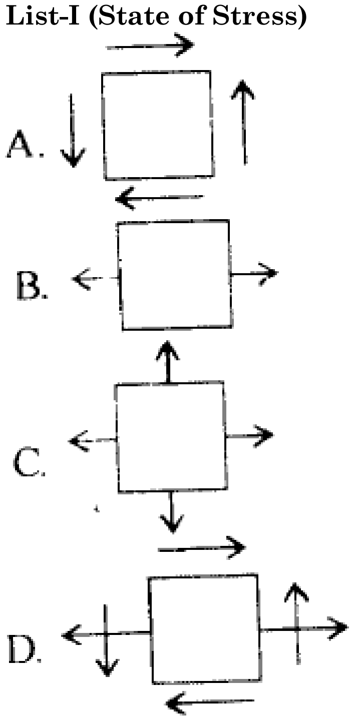
Which of the above statements are correct?

(a) 1 alone (b) 2 alone (c) 1 and 2 (d) 1, 2 and 3

IES-3. Ans. (c)

IES-4. Match List-I with List-II and select the correct answer using the code given below the Lists: [IES-2006]

List-I (State of Stress) **List-II (Kind of Loading)**



1. Combined bending and torsion of circular shaft
2. Torsion of circular shaft
3. Thin cylinder subjected to internal pressure
4. Tie bar subjected to tensile force

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	2	1	3	4	(b)	3	4	2	1
(c)	2	4	3	1	(d)	3	1	2	4

IES-4. Ans. (c)

Section Modulus

IES-5. Two beams of equal cross-sectional area are subjected to equal bending moment. If one beam has square cross-section and the other has circular section, then [IES-1999]

(a) Both beams will be equally strong
(b) Circular section beam will be stronger
(c) Square section beam will be stronger
(d) The strength of the beam will depend on the nature of loading

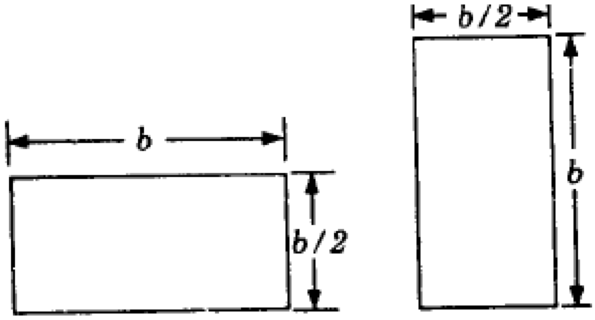
IES-5. Ans. (b) If D is diameter of circle and 'a' the side of square section, $\frac{\pi}{4}d^2 = a^2 \quad or\, d = \sqrt{\frac{4}{\pi}}a$

Z for circular section = $\frac{\pi d^2}{32} = \frac{a^3}{4\sqrt{\pi}}$; and Z for square section = $\frac{a^3}{6}$

IES-6. A beam cross-section is used in two different orientations as shown in the given figure: Bending moments applied to the beam in both cases are same. The maximum bending stresses induced in cases (A) and (B) are related as:

(a) $\sigma_A = 4\sigma_B$ (b) $\sigma_A = 2\sigma_B$

(c) $\sigma_A = \frac{\sigma_B}{2}$ (d) $\sigma_A = \frac{\sigma_B}{4}$



[IES-1997]

IES-6. Ans. (b) Z for rectangular section is $\frac{bd^2}{6}$, $Z_A = \frac{b\left(\frac{b}{2}\right)^2}{6} = \frac{b^3}{24}$, $Z_B = \frac{\frac{b}{2}\times b^2}{6} = \frac{b^3}{12}$

$$M = Z_A.\sigma_A = Z_B.\sigma_B \quad or \frac{b^3}{24}\sigma_A = \frac{b^3}{12}\sigma_B, \quad or\, \sigma_A = 2\sigma_B$$

IES-7. A horizontal beam with square cross-section is simply supported with sides of the square horizontal and vertical and carries a distributed loading that produces maximum bending stress a in the beam. When the beam is placed with one of the diagonals horizontal the maximum bending stress will be: [IES-1993]

(a) $\frac{1}{\sqrt{2}}\sigma$ (b) σ (c) $\sqrt{2}\sigma$ (d) 2σ

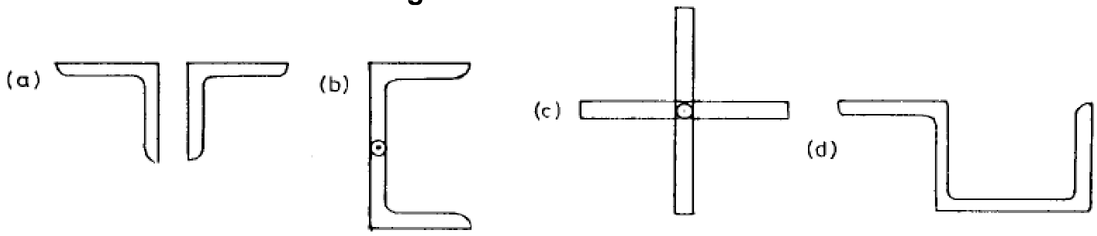
IES-7. Ans. (c) Bending stress = $\frac{M}{Z}$

For rectangular beam with sides horizontal and vertical, Z = $\frac{a^3}{6}$

For same section with diagonal horizontal, Z = $\frac{a^3}{6\sqrt{2}}$

$\therefore$ Ratio of two stresses = $\sqrt{2}$

IES-8. Which one of the following combinations of angles will carry the maximum load as a column? [IES-1994]



IES-8. Ans. (a)

IES-9. Assertion (A): For structures steel I-beams preferred to other shapes. [IES-1992]
Reason (R): In I-beams a large portion of their cross-section is located far from the neutral axis.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-9. Ans. (a)

Combined direct and bending stress

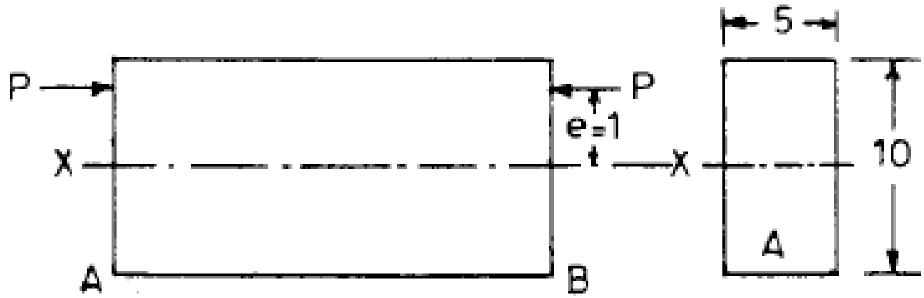
IES-10. Assertion (A): A column subjected to eccentric load will have its stress at centroid independent of the eccentricity. [IES-1994]
Reason (R): Eccentric loads in columns produce torsion.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-10. Ans. (c) A is true and R is false.

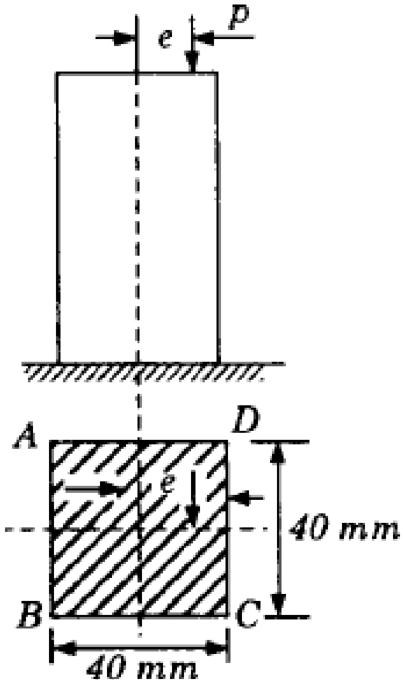
IES-11. For the configuration of loading shown in the given figure, the stress in fibre AB is given by: [IES-1995]

(a) P/A (tensile)
(b) $\left(\dfrac{P}{A}-\dfrac{P.e.5}{I_{xx}}\right)$(Compressive)
(c) $\left(\dfrac{P}{A}+\dfrac{P.e.5}{I_{xx}}\right)$(Compressive)
(d) P/A (Compressive)



IES-11. Ans. (b) $\sigma_d = \dfrac{P}{A}$(compressive), $\sigma_x = \dfrac{My}{I_x} = \dfrac{Pky}{I_x}$(tensile)

IES-12. **A column of square section 40 mm × 40 mm, fixed to the ground carries an eccentric load P of 1600 N as shown in the figure.**
If the stress developed along the edge CD is –1.2 N/mm², the stress along the edge AB will be:



(a) –1.2 N/mm²
(b) +1 N/mm²
(c) +0.8 N/mm²
(d) –0.8 N/mm²

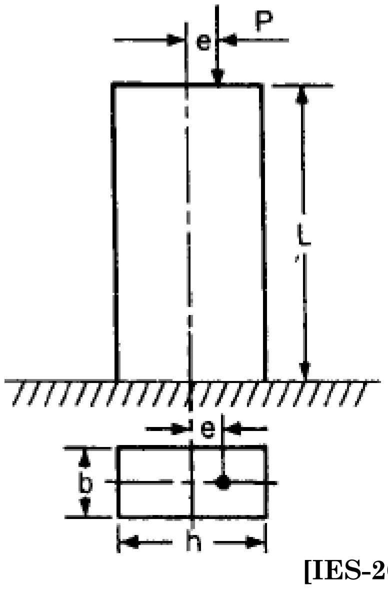
[IES-1999]

IES-12. Ans. (d) Compressive stress at CD = 1.2 N/mm² = $\dfrac{P}{A}\left(1+\dfrac{6e}{b}\right)=\dfrac{1600}{1600}\left(1+\dfrac{6e}{20}\right)$

$$\text{or } \frac{6e}{20}=0.2.\ \text{So stress at } AB=-\frac{1600}{1600}(1-0.2)=-0.8\ \text{N/mm}^2\text{(com)}$$

IES-13. **A short column of symmetric cross-section made of a brittle material is subjected to an eccentric vertical load P at an eccentricity e. To avoid tensile stress in the short column, the eccentricity e should be less than or equal to:**

(a) h/12 (b) h/6
(c) h/3 (d) h/2



[IES-2001]

IES-13. Ans. (b)

IES-14. **A short column of external diameter D and internal diameter d carries an eccentric load W. Toe greatest eccentricity which the load can have without producing tension on the cross-section of the column would be:** **[IES-1999]**

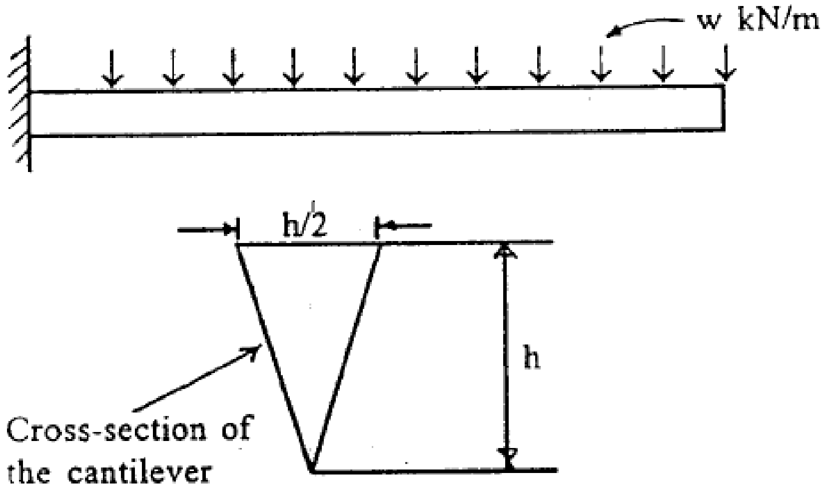
(a) $\dfrac{D+d}{8}$ (b) $\dfrac{D^2+d^2}{8d}$ (c) $\dfrac{D^2+d^2}{8D}$ (d) $\sqrt{\dfrac{D^2+d^2}{8}}$

IES-14. Ans. (c)

Previous 20-Years IAS Questions

Bending equation

IAS-1. Consider the cantilever loaded as shown below: [IAS-2004]



What is the ratio of the maximum compressive to the maximum tensile stress?

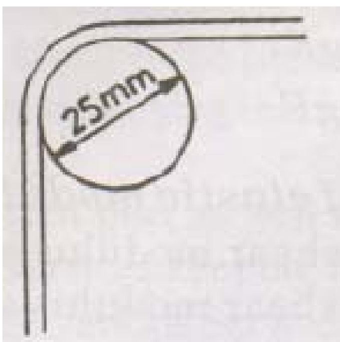
(a) 1.0 (b) 2.0 (c) 2.5 (d) 3.0

IAS-1. Ans. (b) $\sigma = \dfrac{My}{I}$ $\sigma_{\text{compressive, Max}} = \dfrac{M}{I} \times \left(\dfrac{2h}{3}\right)$ at lower end of A.

$$\sigma_{\text{tensile, max}} = \frac{M}{I} \times \left(\frac{h}{3}\right) \text{ at upper end of } B$$

IAS-2. A 0.2 mm thick tape goes over a frictionless pulley of 25 mm diameter. If E of the material is 100 GPa, then the maximum stress induced in the tape is: [IAS 1994]

(a) 100 MPa (b) 200 MPa (c) 400 MPa (d) 800 MPa



IAS-2. Ans. (d) $\dfrac{\sigma}{y} = \dfrac{E}{R}$ Here y = $\dfrac{0.2}{2} = 0.1$ mm = 0.1×10^{-3} m, R = $\dfrac{25}{2}$ mm = 12.5×10^{-3} m

$$\text{or } \sigma = \frac{100 \times 10^3 \times 0.1 \times 10^{-3}}{12.5 \times 10^{-3}} \text{ MPa} = 800\text{MPa}$$

Section Modulus

IAS-3. A pipe of external diameter 3 cm and internal diameter 2 cm and of length 4 m is supported at its ends. It carries a point load of 65 N at its centre. The sectional modulus of the pipe will be: [IAS-2002]

(a) $\frac{65\pi}{64}cm^3$ (b) $\frac{65\pi}{32}cm^3$ (c) $\frac{65\pi}{96}cm^3$ (d) $\frac{65\pi}{128}cm^3$

IAS-3. Ans. (c) Section modulus $(z) = \frac{I}{y} = \frac{\frac{\pi}{64}\left(3^4 - 2^4\right)}{\frac{3}{2}} \text{cm}^3 = \frac{65\pi}{96}cm^3$

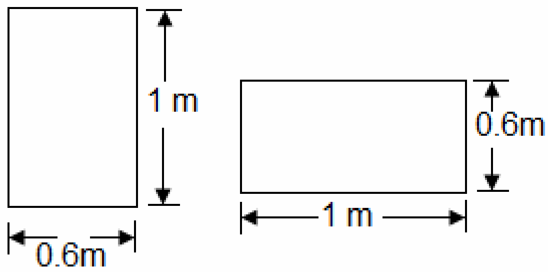
IAS-4. A Cantilever beam of rectangular cross-section is 1m deep and 0.6 m thick. If the beam were to be 0.6 m deep and 1m thick, then the beam would. [IAS-1999]

(a) Be weakened 0.5 times
(b) Be weakened 0.6 times
(c) Be strengthened 0.6 times
(d) Have the same strength as the original beam because the cross-sectional area remains the same

IAS-4. Ans. (b) $z_1 = \frac{I}{y} = \frac{0.6 \times 1^3}{0.5} = 1.2\text{m}^3$

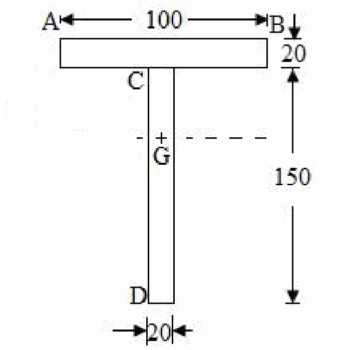
and $z_2 = \frac{I}{y} = \frac{1 \times 0.6^3}{0.3} = 0.72\text{m}^3$

$\therefore \frac{z_2}{z_1} = \frac{0.72}{1.2} = 0.6 \text{ times}$



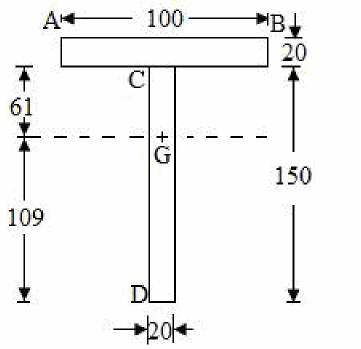
IAS-5. A T-beam shown in the given figure is subjected to a bending moment such that plastic hinge forms. The distance of the neutral axis from D is (all dimensions are in mm)

(a) Zero
(b) 109 mm
(c) 125 mm
(d) 170 mm



[IAS-2001]

IAS-5. Ans. (b)



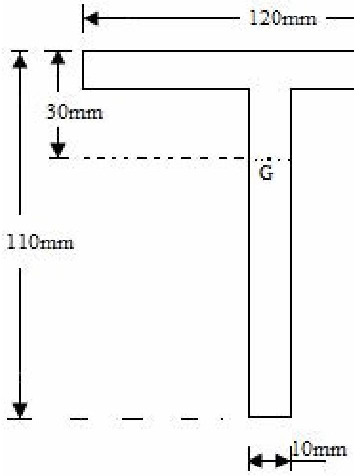
IAS-6. Assertion (A): I, T and channel sections are preferred for beams. [IAS-2000]
Reason(R): A beam cross-section should be such that the greatest possible amount of area is as far away from the neutral axis as possible.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-6. Ans. (a) Because it will increase area moment of inertia, i.e. strength of the beam.

IAS-7. **If the T-beam cross-section shown in the given figure has bending stress of 30 MPa in the top fiber, then the stress in the bottom fiber would be (G is centroid)**

(a) Zero
(b) 30 MPa
(c) –80 MPa
(d) 50 Mpa



[IAS-2000]

IAS-7. Ans. (c) $\frac{M}{I}=\frac{\sigma_1}{y_1}=\frac{\sigma_2}{y_2}\ or\ \sigma_2=y_2\times\frac{\sigma_1}{y_1}=(110-30)\times\frac{30}{30}=80\,MPa$

As top fibre in tension so bottom fibre will be in compression.

IAS-8. **Assertion (A): A square section is more economical in bending than the circular section of same area of cross-section.** **[IAS-1999]**
Reason (R): The modulus of the square section is less than of circular section of same area of cross-section.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-8. ans. (c)

Bimetallic Strip

IAS-9. **A straight bimetallic strip of copper and steel is heated. It is free at ends. The strip, will:** **[IAS-2002]**

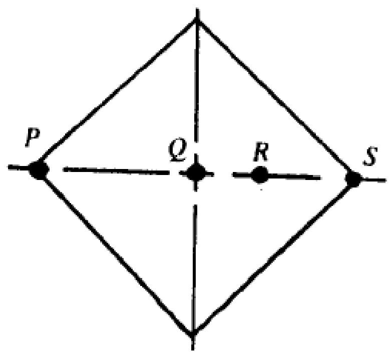
(a) Expand and remain straight
(b) Will not expand but will bend
(c) Will expand and bend also
(d) Twist only

IAS-9. Ans. (c) As expansion of copper will be more than steel.

Combined direct and bending stress

IAS-10. **A short vertical column having a square cross-section is subjected to an axial compressive force, centre of pressure of which passes through point R as shown in the above figure. Maximum compressive stress occurs at point**

(a) S
(b) Q
(c) R
(d) P



[IAS-2002]

IAS-10. Ans. (a) As direct and bending both the stress is compressive here.

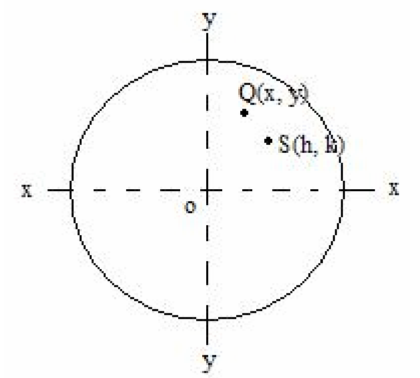
IAS-11. **A strut's cross-sectional area A is subjected to load P a point S (h, k) as shown in the given figure. The stress at the point Q (x, y) is:** **[IAS-2000]**

(a) $\frac{P}{A}+\frac{Phy}{I_x}+\frac{Pkx}{I_y}$

(b) $-\frac{P}{A}-\frac{Phx}{I_y}-\frac{Pky}{I_x}$

(c) $\frac{P}{A}+\frac{Phy}{I_y}+\frac{Pkx}{I_x}$

(d) $\frac{P}{A}+\frac{Phx}{I_y}-\frac{Pky}{I_x}$



IAS-11. Ans. (b) All stress are compressive, direct stress,

$$\sigma_d = \frac{P}{A} \text{ (compressive)}, \ \sigma_x = \frac{My}{I_x} = \frac{Pky}{I_x} \text{ (compressive)}$$

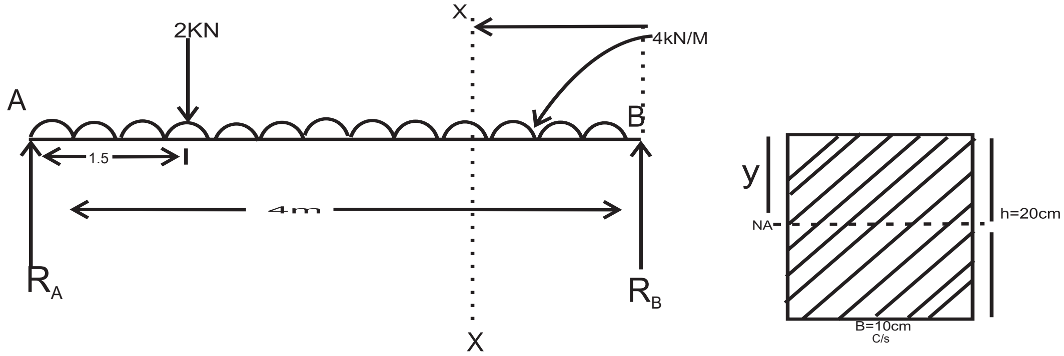
and $\sigma_y = \frac{Mx}{I_y} = \frac{Phx}{I_y}$ (compressive)

Previous Conventional Questions with Answers

Conventional Question IES-2008

Question: **A Simply supported beam AB of span length 4 m supports a uniformly distributed load of intensity q = 4 kN/m spread over the entire span and a concentrated load P = 2 kN placed at a distance of 1.5 m from left end A. The beam is constructed of a rectangular cross-section with width b = 10 cm and depth d = 20 cm. Determine the maximum tensile and compressive stresses developed in the beam to bending.**

Answer:



$R_A + R_B = 2 + 4×4$.........(i)

$-R_A×4 + 2×(4-1.5) + (4×4)×2=0$.......(ii)

or $R_A = 9.25$ kN, $R_B=18-R_A = 8.75$ kN

if $0 \le x \le 2.5$ m

$$M_x=R_B×x - 4x.\left(x/2\right)-2(x-2.5)$$

$=8.75x - 2x^2 - 2x + 5 = 6.75x - 2x^2 + 5$ ...(ii)

From (i) & (ii) we find out that bending movment at x = 2.1875 m in(i) gives maximum bending movement

[Just find $\frac{dM}{dx}$ for both the casses]

$M_{max} = 8.25 \times 2.1875 - 2 \times 1875^2 = 9.57K7kNm$

Area movement of Inertia (I) = $\frac{bh^3}{12} = \frac{0.1 \times 0.2^3}{12} = 6.6667 \times 10^{-5}\ m^4$

Maximum distance from NA is y = 10 cm = 0.1m

$$\sigma_{\max} = \frac{My}{I} = \frac{(9.57 \times 10^3) \times 0.1}{6.6667 \times 10^{-5}} N/m^2 = 14.355MPa$$

Therefore maximum tensile stress in the lowest point in the beam is 14.355 MPa and maximum compressive stress in the topmost fiber of the beam is -14.355 MPa.

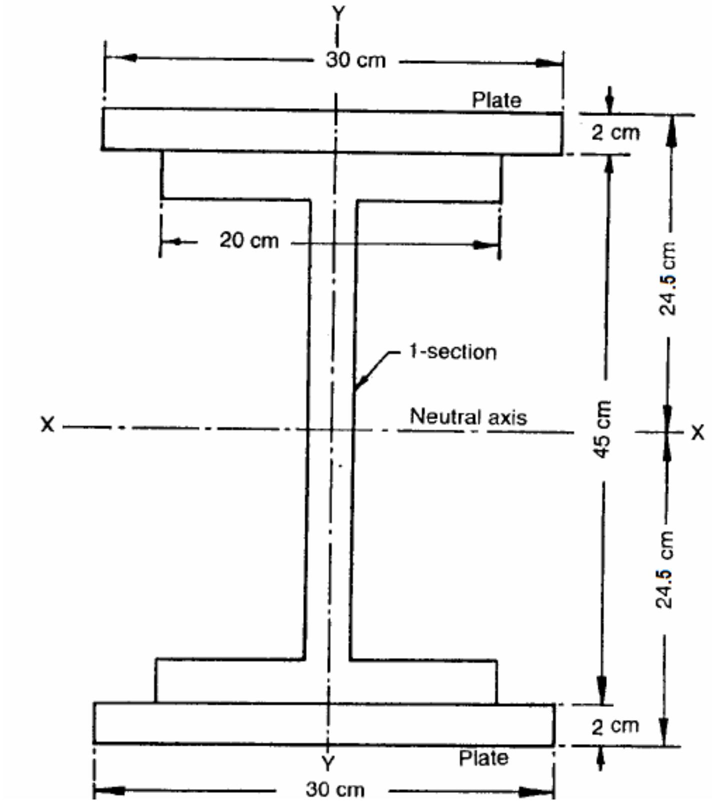
Conventional Question IES-2007

Question: **A simply supported beam made of rolled steel joist (I-section: 450mm × 200mm) has a span of 5 m and it carriers a central concentrated load W. The flanges are strengthened by two 300mm × 20mm plates, one riveted to each flange over the entire length of the flanges. The second moment of area of the joist about the principal bending axis is 35060 cm^4. Calculate**

(i) The greatest central load the beam will carry if the bending stress in the 300mm/20mm plates is not to exceed 125 MPa.

(ii) The minimum length of the 300 mm plates required to restrict the maximum bending stress is the flanges of the joist to 125 MPa.

Answer:



Moment of Inertia of the total section about X-X

(I) = moment of inertia of I –section + moment of inertia of the plates about X-X axis.

$$= 35060 + 2\left[\frac{30\times 2^3}{12} + 30\times 2\times\left(\frac{45}{2}+\frac{2}{2}\right)^2\right] = 101370 \text{ cm}^4$$

(i) Greatest central point load(W):

For a simply supported beam a concentrated load at centre.

$$\text{M} = \frac{\text{WL}}{4} = \frac{W\times 5}{4} = 1.25W$$

$$M = \frac{\sigma.I}{y} = \frac{\left(125\times 10^6\right)\times\left(101370\times 10^{-8}\right)}{0.245} = 517194Nm$$

$\therefore$ 1.25W = 517194 or W = 413.76 kN

(ii) Suppose the cover plates are absent for a distance of x-meters from each support. Then at these points the bending moment must not exceed moment of resistance of 'I' section alone i.e

$$\frac{\sigma.I}{y} = \left(125\times 10^6\right)\times\frac{\left(35060\times 10^{-8}\right)}{0.245} = 178878Nm$$

$\therefore$ *Bending* moment at x metres from each support

$$=\frac{W}{2}\times x = 178878$$

$$or, \frac{41760}{2}\times x = 178878$$

$$or\;\; x = 0.86464\, m$$

Hence leaving 0.86464 m from each support, for the middle 5 - 2×0.86464 = 3.27 m the cover plate should be provided.

Conventional Question IES-2002

***Question*:** **A beam of rectangular cross-section 50 mm wide and 100 mm deep is simply supported over a span of 1500 mm. It carries a concentrated load of 50 kN, 500 mm from the left support.**

Calculate:
(i) The maximum tensile stress in the beam and indicate where it occurs:
(ii) The vertical deflection of the beam at a point 500 mm from the right support. E for the material of the beam = 2 × 10⁵ MPa.

***Answer*:** Taking moment about L

$$R_R \times 1500 = 50\times 500$$

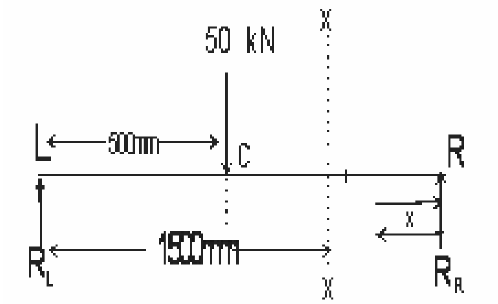
$$or, R_R = 16.667\, kN$$

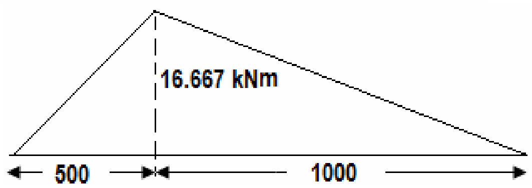
$$or, R_L + R_R = 50$$

$$\therefore R_L = 50 - 16.667 = 33.333 \text{ kN}$$

Take a section from right R, x-x at a distance x.

Bending moment $(M_x) = +R_R . x$





Therefore maximum bending moment will occur at 'c' M_{max}=16.667×1 KNm

(i) Moment of Inertia of beam cross-section

$$(I) = \frac{bh^3}{12} = \frac{0.050\times(0.100)^3}{12} m^4 = 4.1667\times10^{-6} m^4$$

Applying bending equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{\rho} \quad \text{or, } \sigma_{max} = \frac{My}{I} = \frac{\left(16.67\times10^3\right)\times\left(\frac{0.001}{2}\right)}{4.1667\times10^{-6}} N/m^2 = 200\text{MPa}$$

It will occure where M is maximum at point 'C'

(ii) Macaulay's method for determing the deflection of the beam will be convenient as there is point load.

$$M_x = EI\frac{d^2y}{dx^2} = 33.333\times x - 50\times(x-0.5)$$

Integrate both side we get

$$EI\ \frac{d^2y}{dx^2} = 33.333\times\frac{x^2}{2} - \frac{50}{2}(x-0.5)^2 + c_1x + c_2$$

at x=0, y=0 gives $c_2 = 0$

at x=1.5, y=0 gives

$0 = 5.556\times(1.5)^3 - 8.333\times1^3 + c_1\times1.5$

$or, c_1 = -6.945$

$\therefore EIy = 5.556\times x^3 \left|-8.333(x-0.5)^3\right| - 6.945\times1 = -2.43$

$or, y = \dfrac{-2.43}{(2\times10^5\times10^6)\times(4.1667\times10^{-6})}$ m = -2.9167 mm[downward so -ive]

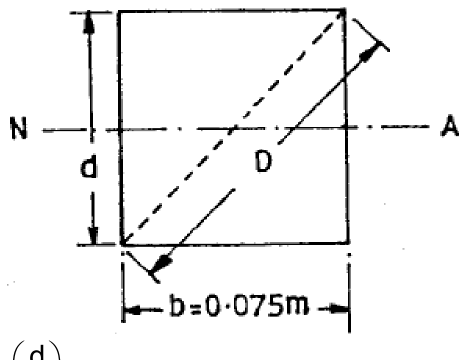
Conventional Question AMIE-1997

Question: **If the beam cross-section is rectangular having a width of 75 mm, determine the required depth such that maximum bending stress induced in the beam does not exceed 40 MN/m²**

Answer: Given: b =75 mm =0·075 m, σ_{max} =40 MN/m²

Depth of the beam, d: Figure below shows a rectangular section of width b = 0·075 m and depth d metres. The bending is considered to take place about the horizontal neutral axis N.A. shown in the figure. The maximum bending stress occurs at the outer fibres of the rectangular section at a distance $\frac{d}{2}$ above or below the neutral axis. Any fibre at a distance y from N.A. is subjected to a bending stress, $\sigma = \frac{My}{I}$, where I denotes the second moment of area of the rectangular section about the N.A. i.e. $\frac{bd^3}{12}$.

At the outer fibres, y = $\frac{d}{2}$, the maximum bending stress there becomes



$$\sigma_{max} = \frac{M\times\left(\frac{d}{2}\right)}{\frac{bd^3}{12}} = \frac{M}{\frac{bd^2}{6}} \qquad ----(i)$$

or $$M = \sigma_{max}.\frac{bd^2}{6} \qquad ----(ii)$$

For the condition of maximum strength i.e. maximum moment M, the product bd² must be a maximum, since σ_{max} is constant for a given material. To maximize the quantity bd² we realise that it must be expressed in terms of one independent variable, say, b, and we may do this from the right angle triangle relationship.

$$b^2 + d^2 = D^2$$

or $$d^2 = D^2 - b^2$$

Multiplying both sides by b, we get $bd^2 = bD^2 - b^3$

To maximize bd² we take the first derivative of expression with respect to b and set it equal to zero, as follows:

$$\frac{d}{db}\left(bd^2\right) = \frac{d}{db}\left(bD^2 - b^3\right) = D^2 - 3b^2 = b^2 + d^2 - 3b^2 = d^2 - 2b^2 = 0$$

Solving, we have, depth d $\sqrt{2}$ b ...(iii)

This is the desired radio in order that the beam will carry a maximum moment M.

It is to be noted that the expression appearing in the denominator of the right side of eqn. (i) i. e. $\frac{bd^2}{6}$ is the section modulus (Z) of a rectangular bar. Thus, it follows; the section modulus is actually the quantity to be maximized for greatest strength of the beam.

Using the relation (iii), we have

d = $\sqrt{2}$ x 0·075 = 0·0106 m

Now, M = σ_{max} x Z = σ_{max} x $\frac{bd^2}{6}$

Substituting the values, we get

M = 40 × $\frac{0.075 \times (0.106)^2}{6}$ = 0.005618 MNm

$$\sigma_{max} = \frac{M}{Z} = \frac{0.005618}{\left(0.075 \times (0.106)2/6\right)} = 40\,MN/m^2$$

Hence, the required depth d = 0·106 m = 106 mm

7. Shear Stress in Beam

Theory at a Glance (for IES, GATE, PSU)

1. Shear stress in bending (τ)

$$\tau = \frac{vQ}{Ib}$$

Where, V = Shear force = $\dfrac{dM}{dx}$

Q = Statical moment = $\displaystyle\int_{y_1}^{c_1} ydA$

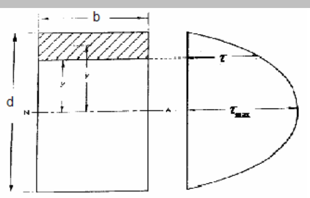
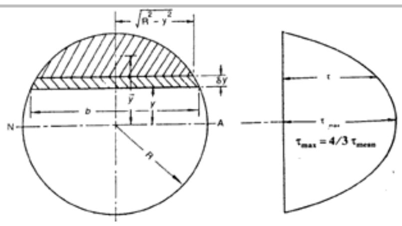
I = Moment of inertia

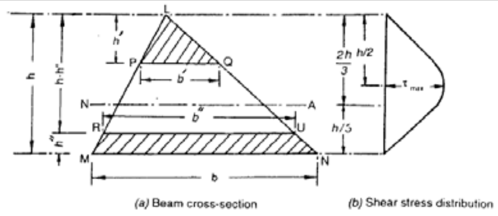
b = Width of beam c/s.

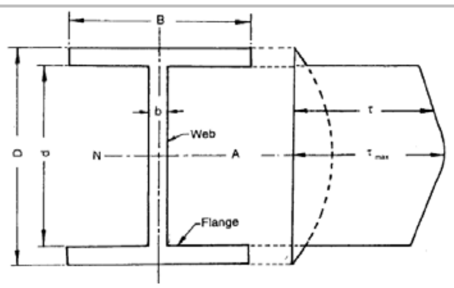
2. Statical Moment (Q)

Q= $\displaystyle\int_{y_1}^{c_1} ydA$ = Shaded Area × distance of the centroid of the shaded area from the neutral axis of the c/s.

3. Variation of shear stress

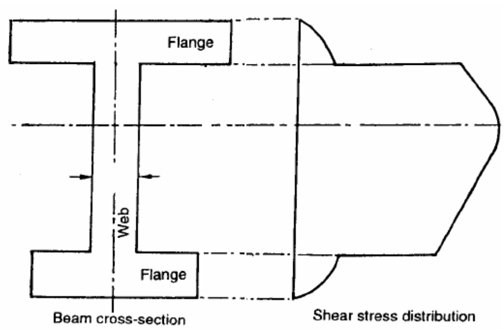
Section	Diagram	Position of τ_{max}	τ_{max}
Rectangular		N.A	$\tau_{max} = \dfrac{3V}{2A}$ $\tau_{max} = 1.5\tau_{mean}$ $= \tau_{NA}$
Circular		N.A	$\tau_{max} = \dfrac{4}{3}\tau_{mean}$

Triangular		$\frac{h}{6}$ from N.A	$\tau_{\max} = 1.5\tau_{mean}$ $\tau_{NA} = 1.33\,\tau_{mean}$
Trapezoidal		$\frac{h}{6}$ from N.A	

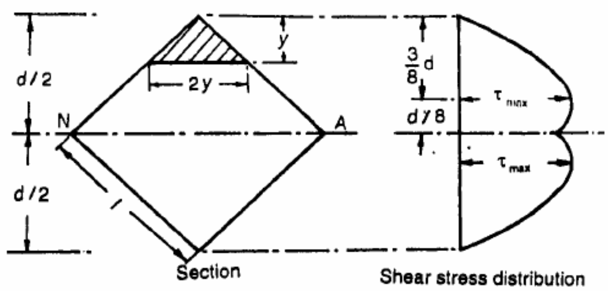
Section	Diagram	$\tau_{\max}$
Uni form I-Section		In Flange, $(\tau \max)\left(\tau_{\max}\right)_{y_1=\frac{h_1}{2}} = \frac{V}{8I}\left[h^{2-h1^2}\right]$ $\left(\tau_{\max}\right)_{y_1=h/2} = o$ In Web $\left(\tau_{\max}\right)_{y_1=o} = \frac{v}{8It}\left[b(h_1^{2-}h_1^{2}) + th_1^{2}\right]$ $\left(\tau_{mim}\right)_{y_1=\frac{h1}{2}} = \frac{vb}{8It}\left[h^2 - h_1^{2}\right]$

4. Variation of shear stress for some more section [Asked in different examinations]

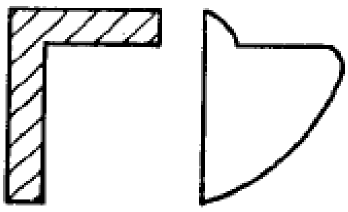
Non uniform I-Section



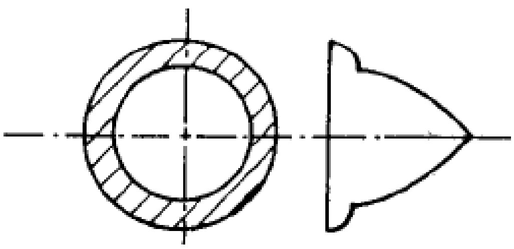
Diagonally placed square section



L-section

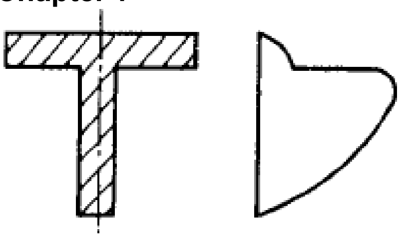


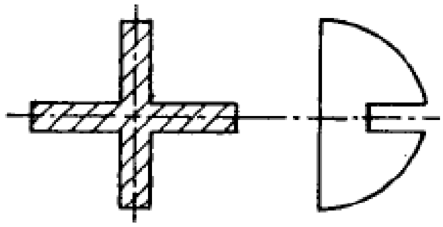
Hollow circle



T-section

Cross



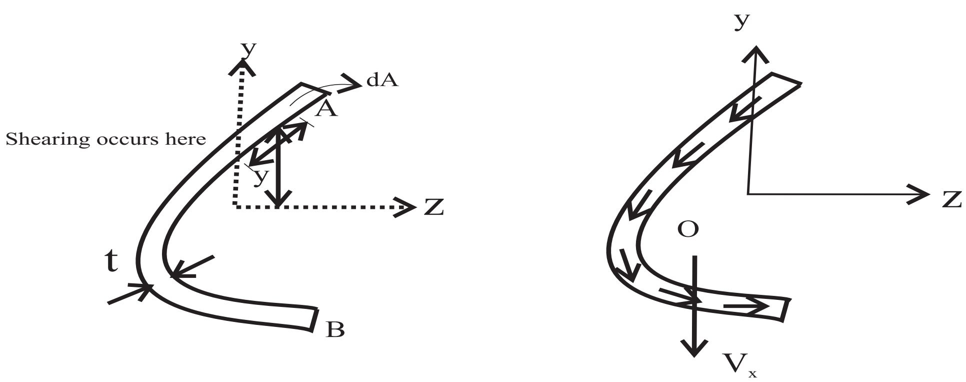


5. Rectangular section

- Maximum shear stress for rectangular beam: $\tau_{max} = \dfrac{3V}{2A}$
- For this, A is the area of the entire cross section
- Maximum shear occurs at the neutral axis
- Shear is zero at the top and bottom of beam

6. Shear stress in beams of thin walled profile section.

- **Shear stress at any point in the wall distance "s" from the free edge**



$$\tau = \frac{V_x}{It}\int_o^s y\,dA$$

$where\ V_x = Shear$ force

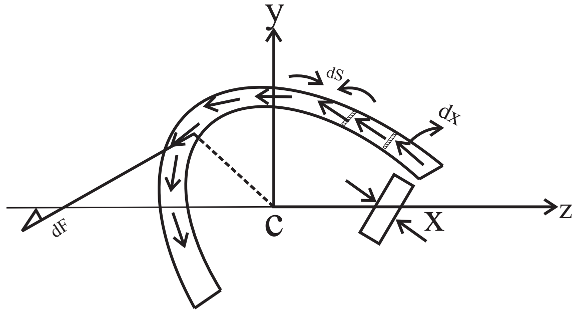
τ = Thickness of the section

I = Moment of inrertia about NA

- **Shear Flow (q)**

q = $\tau t = \dfrac{V_x}{I_{NA}}\displaystyle\int_o^s y\,dA$

- **Shear Force (F)**



F= $\int qds$

- **Shear Centre (e)**

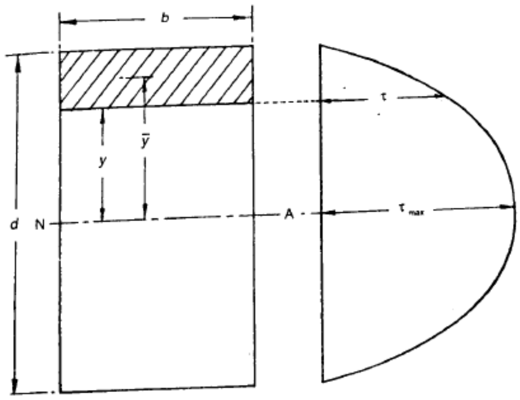
  Point of application of shear stress resultant

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Shear Stress Variation

GATE-1. The transverse shear stress acting in a beam of rectangular cross-section, subjected to a transverse shear load, is:
(a) Variable with maximum at the bottom of the beam
(b) Variable with maximum at the top of the beam
(c) Uniform
(d) Variable with maximum on the neutral axis



[IES-1995, GATE-2008]

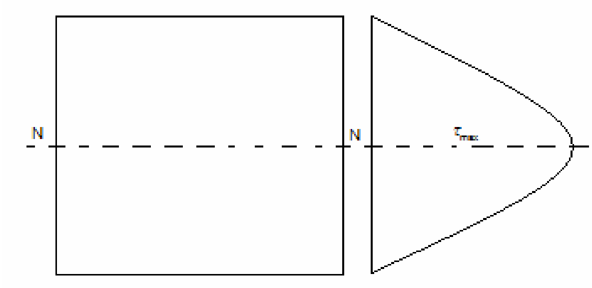
GATE-1. Ans (d) $\tau_{\max} = \frac{3}{2}\tau_{mean}$

GATE-2. The ratio of average shear stress to the maximum shear stress in a beam with a square cross-section is: [GATE-1994, 1998]

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 2

GATE-2. Ans. (b)

$$\tau_{\max} = \frac{3}{2}\tau_{\text{mean}}$$



Previous 20-Years IES Questions

Shear Stress Variation

IES-1. At a section of a beam, shear force is F with zero BM. The cross-section is square with side a. Point A lies on neutral axis and point B is mid way between neutral axis and top edge, i.e. at distance a/4 above the neutral axis. If τ_A and τ_B denote shear stresses at points A and B, then what is the value of τ_A / τ_B? [IES-2005]

(a) 0 (b) ¾ (c) 4/3 (d) None of above

IES-1. Ans. (c) $\tau = \frac{VA\bar{y}}{Ib} = \frac{V \times \frac{a}{2}\left(\frac{a^2}{4} - y^2\right)}{\frac{a^4}{12} \times a} = \frac{3}{2}\frac{V}{a^3}\left(a^2 - 4y^2\right)$ or $\frac{\tau_A}{\tau_B} = \frac{\frac{3}{2}\frac{V}{a^3}.a^2}{\frac{3}{2}.\frac{V}{a^3}.\left\{\left(a^2 - 4\left(\frac{a}{4}\right)^2\right)\right\}} = \frac{4}{3}$

IES-2. **A wooden beam of rectangular cross-section 10 cm deep by 5 cm wide carries maximum shear force of 2000 kg. Shear stress at neutral axis of the beam section is:** **[IES-1997]**

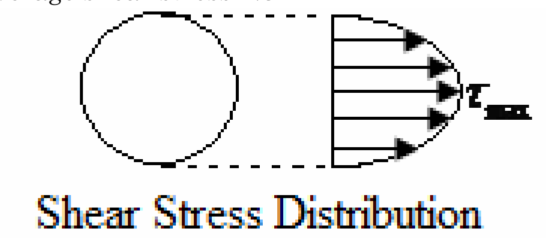
(a) Zero (b) 40 kgf/cm² (c) 60 kgf/cm² (d) 80 kgf/cm²

IES-2. Ans. (c) Shear stress at neutral axis = $\frac{3}{2}\times\frac{F}{bd}=\frac{3}{2}\times\frac{2000}{10\times5}=60\,\text{kg/cm}^2$

IES-3. **In case of a beam of circular cross-section subjected to transverse loading, the maximum shear stress developed in the beam is greater than the average shear stress by:** **[IES-2006; 2008]**

(a) 50% (b) 33% (c) 25% (d) 10%

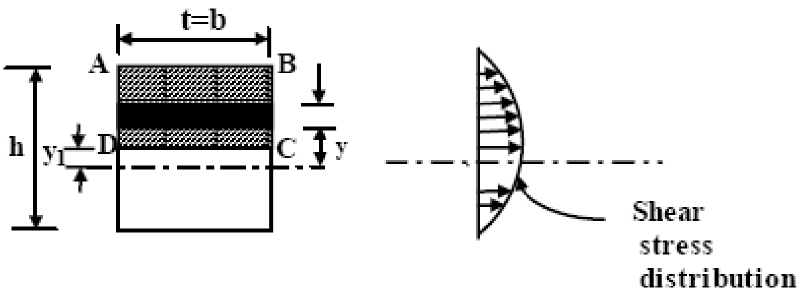
IES-3. Ans. (b) In the case of beams with circular cross-section, the ratio of the maximum shear stress to average shear stress 4:3



IES-4. What is the nature of distribution of shear stress in a rectangular beam? **[IES-1993, 2004; 2008]**

(a) Linear (b) Parabolic (c) Hyperbolic (d) Elliptic

IES-4. Ans. (b)

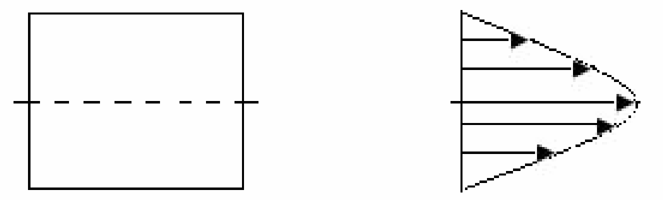


$\tau=\frac{V}{4I}\left(\frac{h^2}{4}-y_1^2\right)$ indicating a parabolic distribution of shear stress across the cross-section.

IES-5. **Which one of the following statements is correct?** **[IES 2007]**
When a rectangular section beam is loaded transversely along the length, shear stress develops on

(a) Top fibre of rectangular beam (b) Middle fibre of rectangular beam
(c) Bottom fibre of rectangular beam (d) Every horizontal plane

IES-5. Ans. (b)

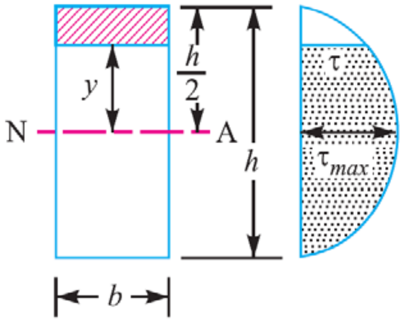


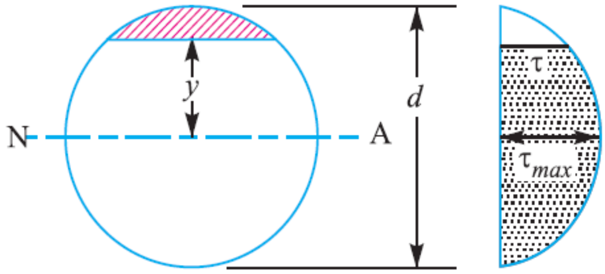
IES-6. **A beam having rectangular cross-section is subjected to an external loading. The average shear stress developed due to the external loading at a particular**

cross-section is t_{avg}. What is the maximum shear stress developed at the same cross-section due to the same loading? **[IES-2009]**

(a) $\frac{1}{2}t_{avg}$ (b) t_{avg} (c) $\frac{3}{2}t_{avg}$ (d) $2\,t_{avg}$

IES-6. Ans. (c)





Shear stress in a rectangular beam, maximum shear stress,

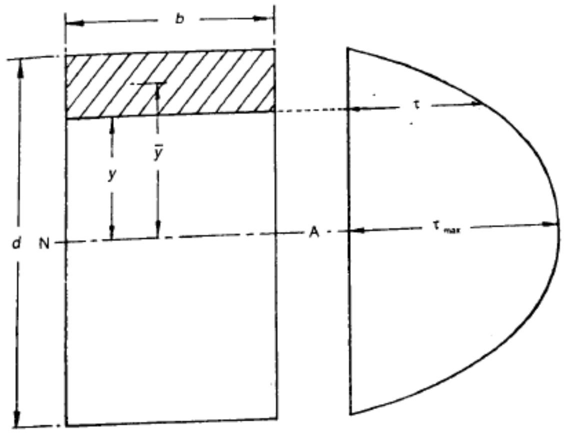
$$\tau_{max} = \frac{3F}{2b.h} = 1.5\ \tau_{(average)}$$

Shear stress in a circular beam, the maximum shear stress,

$$\tau_{max} = \frac{4F}{3 \times \frac{\pi}{4}d^2} = \frac{4}{3}\tau_{(average)}$$

IES-7. The transverse shear stress acting in a beam of rectangular cross-section, subjected to a transverse shear load, is:

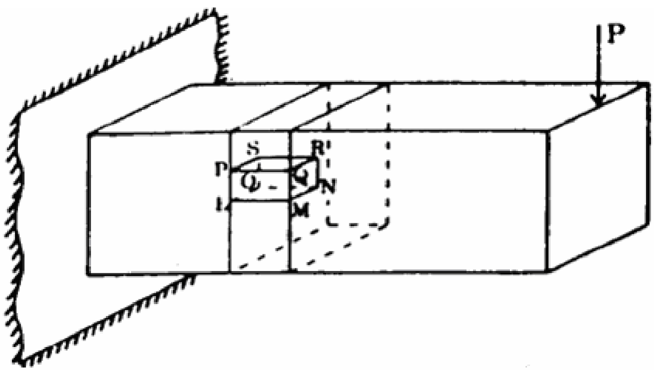
(a) Variable with maximum at the bottom of the beam
(b) Variable with maximum at the top of the beam
(c) Uniform
(d) Variable with maximum on the neutral axis



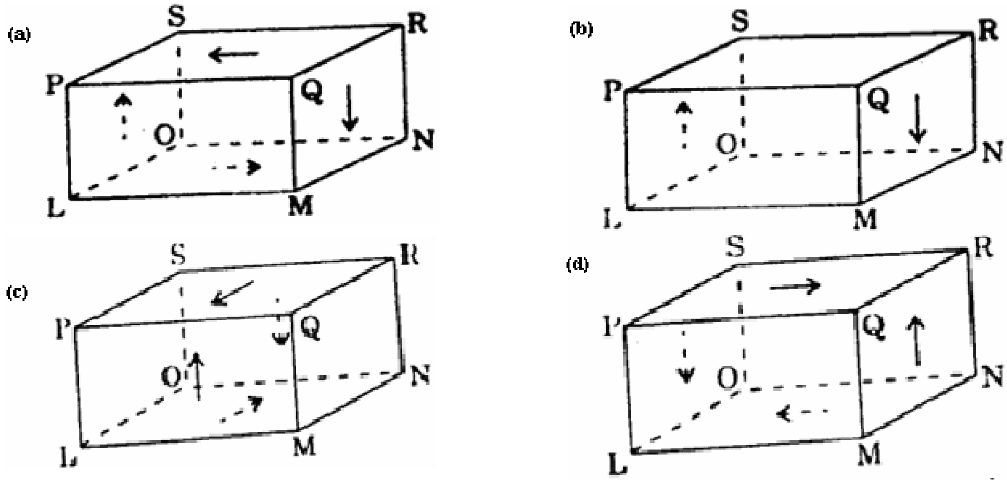
[IES-1995, GATE-2008]

IES-7. Ans (d) $\tau_{\max} = \frac{3}{2}\tau_{mean}$

IES-8.



A cantilever is loaded by a concentrated load P at the free end as shown. The shear stress in the element LMNOPQRS is under consideration. Which of the following figures represents the shear stress directions in the cantilever? **[IES-2002]**



IES-8. Ans. (d)

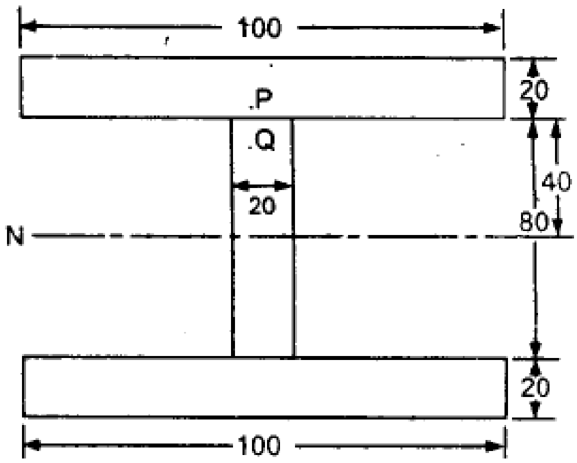
IES-9. In I-Section of a beam subjected to transverse shear force, the maximum shear stress is developed. [IES- 2008]

(a) At the centre of the web (b) At the top edge of the top flange
(c) At the bottom edge of the top flange (d) None of the above

IES-9. Ans. (a)

IES-10. The given figure (all dimensions are in mm) shows an I-Section of the beam. The shear stress at point P (very close to the bottom of the flange) is 12 MPa. The stress at point Q in the web (very close to the flange) is:

(a) Indeterminable due to incomplete data
(b) 60MPa
(c) 18 MPa
(d) 12 MPa



[IES-2001]

IES-10. Ans. (b)

IES-11. Assertion (A): In an I-Section beam subjected to concentrated loads, the shearing force at any section of the beam is resisted mainly by the web portion.
Reason (R): Average value of the shearing stress in the web is equal to the value of shearing stress in the flange. [IES-1995]

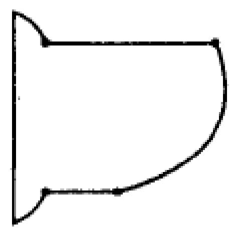
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-11. Ans. (c)

Shear stress distribution for different section

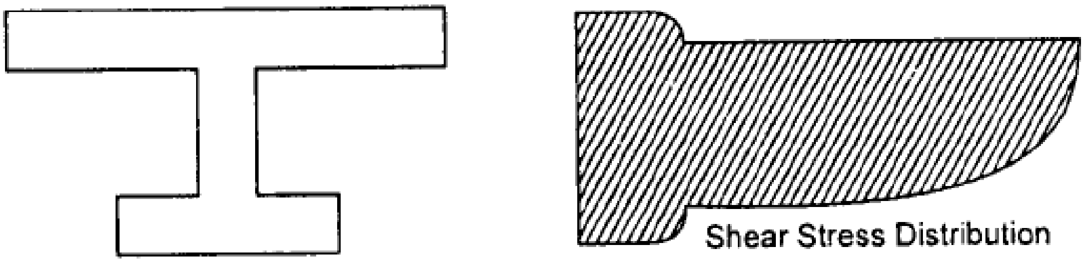
IES-12. The shear stress distribution over a beam cross-section is shown in the figure above. The beam is of

(a) Equal flange I-Section
(b) Unequal flange I-Section
(c) Circular cross-section
(d) T-section



[IES-2003]

IES-12. Ans. (b)



Previous 20-Years IAS Questions

Shear Stress Variation

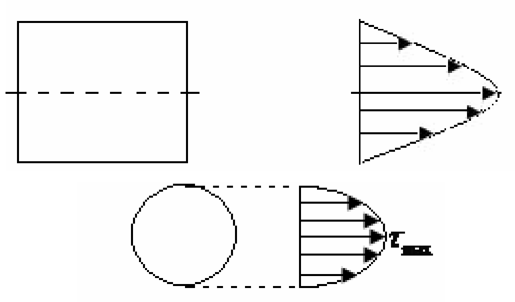
IAS-1. Consider the following statements: [IAS-2007]
Two beams of identical cross-section but of different materials carry same bending moment at a particular section, then
1. **The maximum bending stress at that section in the two beams will be same.**
2. **The maximum shearing stress at that section in the two beams will be same.**
3. **Maximum bending stress at that section will depend upon the elastic modulus of the beam material.**
4. **Curvature of the beam having greater value of E will be larger.**

Which of the statements given above are correct?
(a) 1 and 2 only (b) 1, 3 and 4 (c) 1, 2 and 3 (d) 2, 3 and 4

IAS-1. Ans. (a) Bending stress $\delta = \frac{My}{I}$ and shear stress $(\tau) = \frac{VA\bar{y}}{Ib}$ both of them does not depends on material of beam.

IAS-2. In a loaded beam under bending [IAS-2003]
(a) Both the maximum normal and the maximum shear stresses occur at the skin fibres
(b) Both the maximum normal and the maximum shear stresses occur the neutral axis
(c) The maximum normal stress occurs at the skin fibres while the maximum shear stress occurs at the neutral axis
(d) The maximum normal stress occurs at the neutral axis while the maximum shear stress occurs at the skin fibres

IAS-2. Ans. (c)

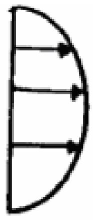


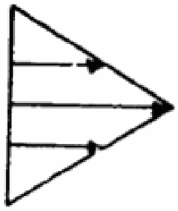
Shear Stress Distribution

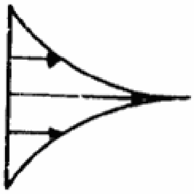
$\tau = \frac{V}{4I}\left(\frac{h^2}{4} - y_1^2\right)$ indicating a parabolic distribution of shear stress across the cross-section.

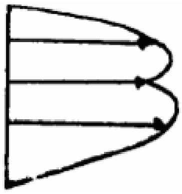
Shear stress distribution for different section

IAS-3. Select the correct shear stress distribution diagram for a square beam with a diagonal in a vertical position: [IAS-2002]

(a) 

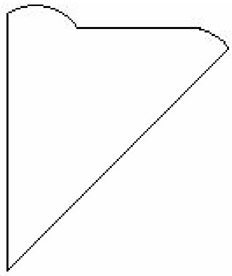
(b) 

(c) 

(d) 

IAS-3. Ans. (d)

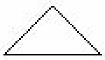
IAS-4. The distribution of shear stress of a beam is shown in the given figure. The cross-section of the beam is: [IAS-2000]



(a) 1

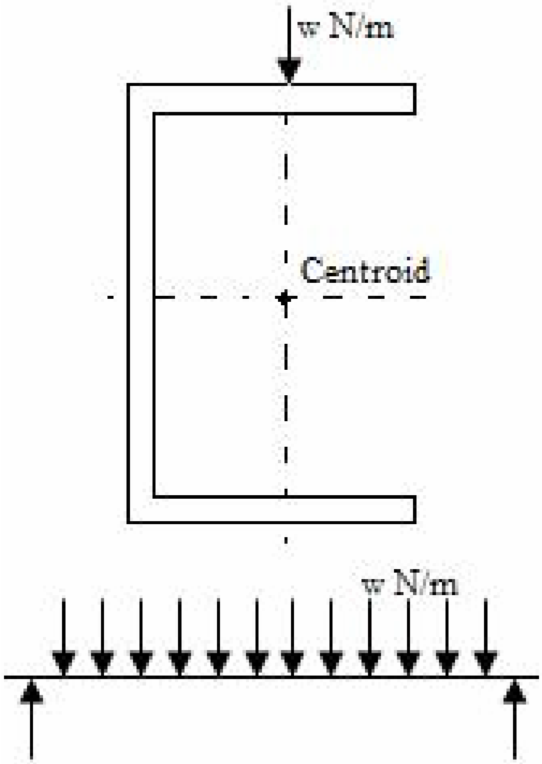
(b) T

(c) 

(d) 

IAS-4. Ans. (b)

IAS-5. A channel-section of the beam shown in the given figure carries a uniformly distributed load. [IAS-2000]



Assertion (A): The line of action of the load passes through the centroid of the cross-section. The beam twists besides bending.

Reason (R): Twisting occurs since the line of action of the load does not pass through the web of the beam.

(a) Both A and R are individually true and R is the correct explanation of A

(b) Both A and R are individually true but R is **NOT** the correct explanation of A

(c) A is true but R is false
(d) A is false but R is true

IAS-5. Ans. (c) Twisting occurs since the line of action of the load does not pass through the shear.

Previous Conventional Questions with Answers

Conventional Question IES-2009

Q. **(i)A cantilever of circular solid cross-section is fixed at one end and carries a concentrated load P at the free end. The diameter at the free end is 200 mm and increases uniformly to 400 mm at the fixed end over a length of 2 m. At what distance from the free end will the bending stresses in the cantilever be maximum? Also calculate the value of the maximum bending stress if the concentrated load P = 30 kN** **[15-Marks]**

Ans. We have $\frac{\sigma}{y} = \frac{M}{I}$ .... (i)

Taking distance x from the free end we have

$$M = 30x \text{ kN.m} = 30x \times 10^3 \text{ N.m}$$

$$y = 100 + \frac{x}{2}(200 - 100)$$

$$= 100 + 50x \text{ mm}$$

$$\text{and } I = \frac{\pi d^4}{64}$$

Let d be the diameter at x from free end.

$$= \frac{\pi\left[200 + \frac{(400-200)}{2}x\right]^4}{64}$$

$$= \frac{\pi(200+100x)^4}{64} \text{ mm}^4$$

From equation (i), we have

$$\frac{\sigma}{(100+50x)\times 10^{-3}} = \frac{30x \times 10^3}{\frac{\pi}{64}(200+100x)^4 \times 10^{-12}}$$

$$\therefore \sigma = \frac{960x}{\pi}(200+100x)^{-3} \times 10^{12} \quad \text{...... (ii)}$$

$$= \frac{960x}{\pi}(200+100x)^{-3} \times 10^{12}$$

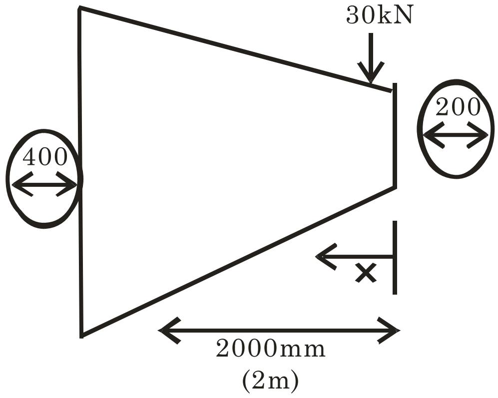
$$\text{For max } \sigma, \frac{d\sigma}{dx} = 0$$

$$\therefore \frac{10^{12} \times 960}{\pi}$$

$$\left[x(-3)(100)(200+100x)^{-4} + 1.(200+100x)^{-3}\right] = 0$$

$$\Rightarrow -300x + 200 + 100x = 0$$

$$\Rightarrow \boxed{x = 1m}$$



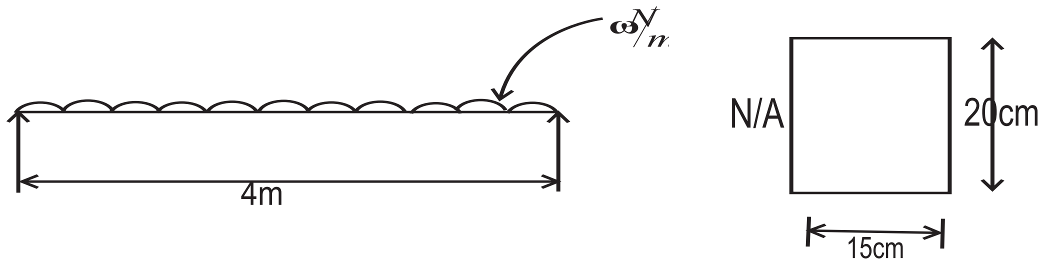
Hence maximum bending stress occurs at the midway and from equation (ii), maximum bending stress

$$\sigma = \frac{960}{\pi}(1)(200+100)^{-3} \times 10^{12}$$

$$= \frac{960 \times 10^{12}}{\pi \times (300)^3} = 11.32 \text{ MPa}$$

Conventional Question IES-2006

Question: **A timber beam 15 cm wide and 20 cm deep carries uniformly distributed load over a span of 4 m and is simply supported.**
If the permissible stresses are 30 N/mm² longitudinally and 3 N/mm² transverse shear, calculate the maximum load which can be carried by the timber beam.



Answer: Moment of inertia (I) $= \frac{bh^3}{12} = \frac{(0.15)\times(0.20)^3}{12} = 10^{-4}\text{ m}^4$

Distance of neutral axis from the top surface $y = \frac{20}{2} = 10\text{cm} = 0.1\text{ m}$

We know that $\frac{M}{I} = \frac{\sigma}{y}$ or $\sigma = \frac{My}{I}$

Where maximum bending moment due to uniformly distributed load in simply supported beam $(M) = \frac{\omega\ell^2}{8} = \frac{\omega \times 4^2}{8} = 2\omega$

Considering longitudinal stress

$$30 \times 10^6 = \frac{(2\omega)\times 0.1}{10^{-4}}$$

or, $\omega = 15\text{ kN/m}$

Now consideng Shear

Maximum shear force $= \frac{\omega.L}{2} = \frac{\omega.4}{2} = 2\omega$

Therefore average shear stress $(\tau_{\text{mean}}) = \frac{2\omega}{0.15 \times 0.2} = 66.67\,\omega$

For rectangular cross-section

Maximum shear stress$(\tau_{\max}) = \frac{3}{2}\,.\,\tau = \frac{3}{2} \times 66.67\omega = 100\,\omega$

Now $3 \times 10^6 = 100\omega$; $\omega = 30$ kN/m

So maximum load carring capacity of the beam = 15 kN/m (without fail).

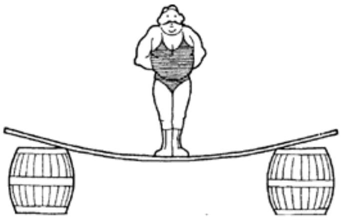
8. Fixed and Continuous Beam

Theory at a Glance (for IES, GATE, PSU)

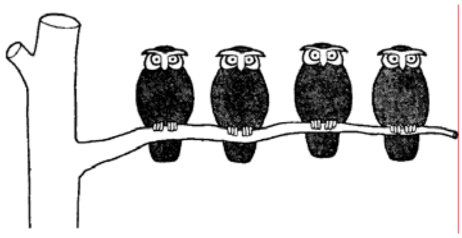
What is a beam?

A (usually) horizontal structural member that is subjected to a load that tends to bend it.

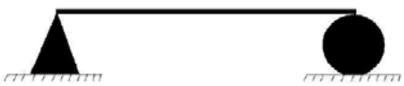
Types of Beams



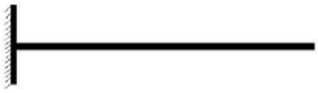
Simply supported beam



Cantilever beam



Simply Supported Beams



Cantilever Beam

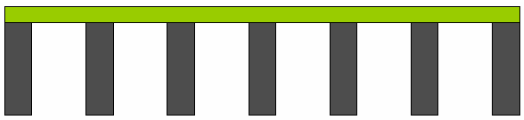
Continuous Beam

Single Overhang Beam

Double Overhang Beam

Single Overhang Beam with internal hinge

Fixed Beam



Continuous beam

Continuous beams

Beams placed on more than 2 supports are called continuous beams. Continuous beams are used when the span of the beam is very large, deflection under each rigid support will be equal zero.

Analysis of Continuous Beams

(Using 3-moment equation)

Stability of structure

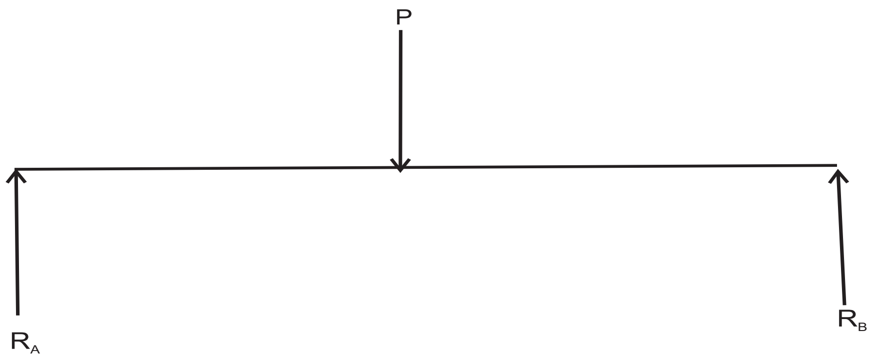
If the equilibrium and geometry of structure is maintained under the action of forces than the structure is said to be stable.

External stability of the structure is provided by the reaction at the supports. Internal stability is provided by proper design and geometry of the member of the structure.

Statically determinate and indeterminate structures

Beams for which reaction forces and internal forces can be found out from static equilibrium equations alone are called statically determinate beam.

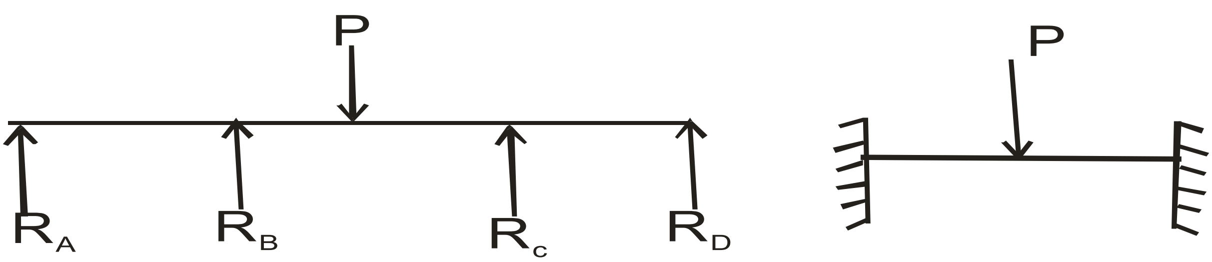
Example:



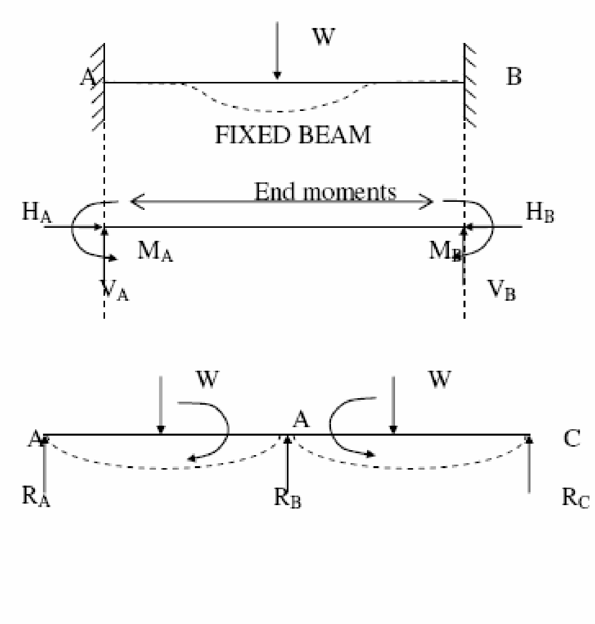
$\sum X_i = 0, \sum Y_i = 0$ and $\sum M_i = 0$ is sufficient to calculate R_A & R_B.

Beams for which reaction forces and internal forces **cannot** be found out from static equilibrium equations alone are called statically indeterminate beam. This type of beam requires deformation equation in addition to static equilibrium equations to solve for unknown forces.

Example:



Ex:



No. of unknowns = 6

No. of eq . Condition = 3

Therefore statically indeterminate

Degree of indeterminacy =6-3 = 3

No. of unknowns = 3

No. of equilibrium Conditions = 2

Therefore Statically indeterminate

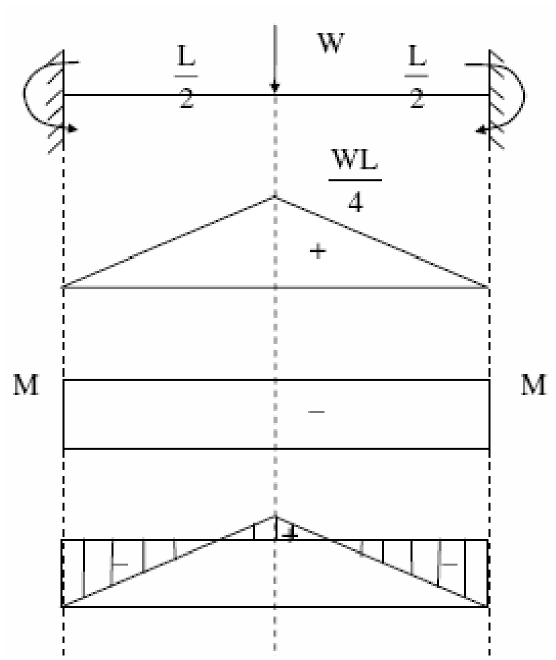
Degree of indeterminacy = 1

Advantages of fixed ends or fixed supports

- Slope at the ends is zero.
- Fixed beams are stiffer, stronger and more stable than SSB.
- In case of fixed beams, fixed end moments will reduce the BM in each section.
- The maximum deflection is reduced.

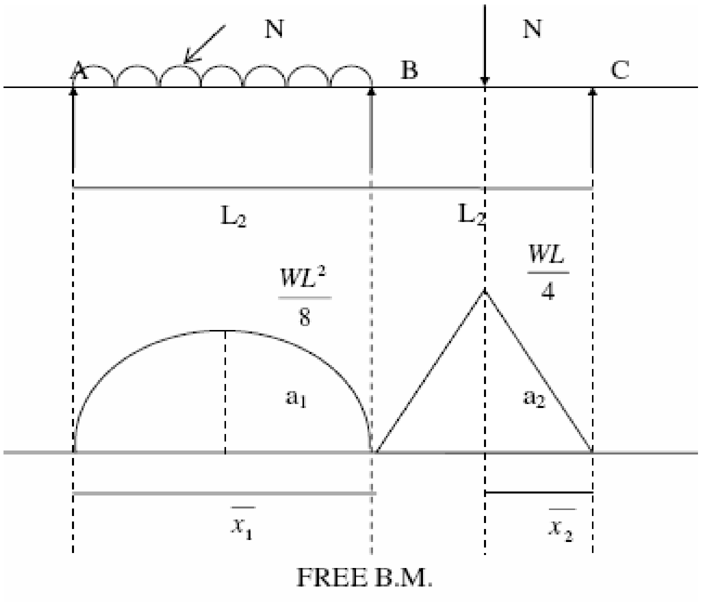
Bending moment diagram for fixed beam

Example:



BMD for Continuous beams

BMD for continuous beams can be obtained by superimposing the fixed end moments diagram over the free bending moment diagram.



Three - moment Equation for continuous beams OR

Clapeyron's Three Moment Equation

$$M_A\left(\frac{L_1}{E_1 I_1}\right)+2M_B\left(\frac{L_1}{E_1 I_1}+\frac{L_2}{E_2 I_2}\right)+M_C\left(\frac{L_2}{E_2 I_2}\right)$$

$$=\frac{-6a_1\overline{x_1}}{E_1 I_1 L_1}-\frac{6a_2\overline{x_2}}{E_2 I_2 L_2}-6\left[\frac{\delta_A-\delta_B}{L_1}+\frac{\delta_C-\delta_B}{L_2}\right]$$

The above equation is called generalized 3-moments Equation.

M_A, M_B and M_C are support moments E_1, E_2 $\rightarrow$ Young's modulus
of Elasticity of 2 spans.

I_1, I_2 $\rightarrow$ M O I of 2 spans,

a_1, a_2 $\rightarrow$ Areas of free B.M.D.

$\overline{x_1}$ and $\overline{x_2}$ $\rightarrow$ Distance of free B.M.D. from the end supports, or outer supports.
(A and C)

δ_A, δ_B and δ_C $\rightarrow$ are sinking or settlements of support from their initial position.

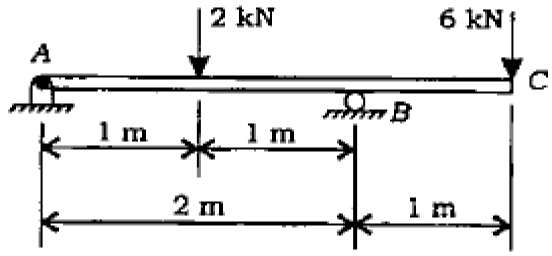
OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years IES Questions

Overhanging Beam

IES-1. An overhanging beam ABC is supported at points A and B, as shown in the above figure. Find the maximum bending moment and the point where it occurs. [IES-2009]

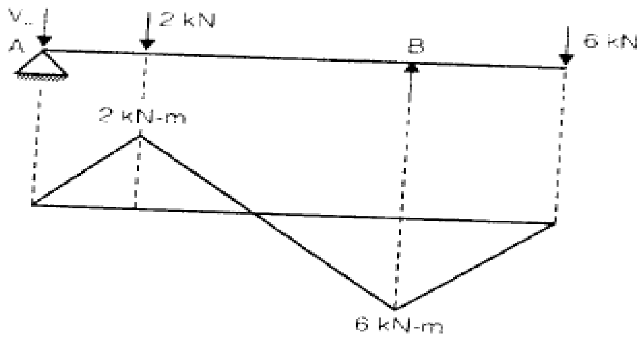
(a) 6 kN-m at the right support
(b) 6 kN-m at the left support
(c) 4.5 kN-m at the right support
(d) 4.5 kN-m at the midpoint between the supports



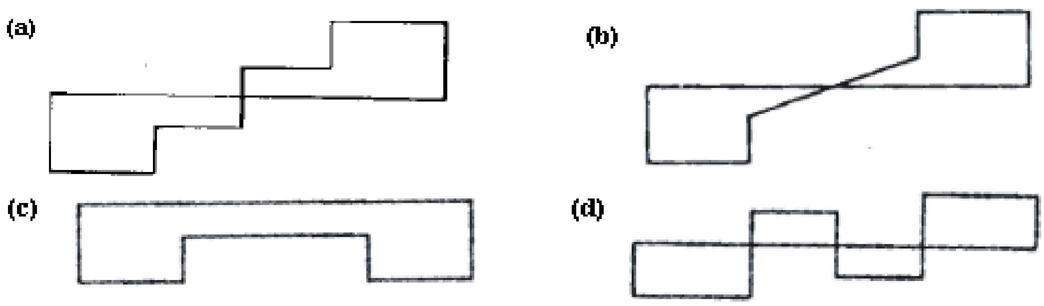
IES-1. Ans. (a) Taking moment about A

$$V_B \times 2 = (2\times 1) + (6\times 3)$$
$$\Rightarrow \quad 2V_B = 2 + 18$$
$$\Rightarrow \quad V_B = 10 \text{ kN}$$
$$V_A + V_B = 2 + 6 = 8\text{kN}$$
$$\therefore \quad V_A = 8 - 10 = -2 \text{ kN}$$

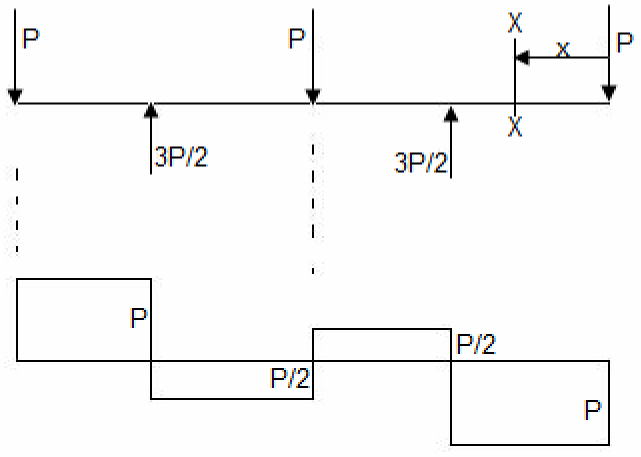
$\therefore$ Maximum Bending Moment = 6 kN-m at the right support



IES-2. A beam of length 4 L is simply supported on two supports with equal overhangs of L on either sides and carries three equal loads, one each at free ends and the third at the mid-span. Which one of the following diagrams represents correct distribution of shearing force on the beam? [IES-2004]

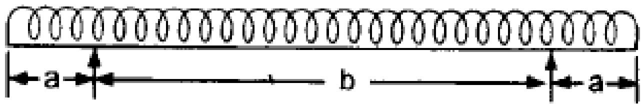


IES-2. Ans. (d)



They use opposite sign conversions but for correct sign remember S.F & B.M of cantilever is (-) ive.

IES-3. A horizontal beam carrying uniformly distributed load is supported with equal overhangs as shown in the given figure



The resultant bending moment at the mid-span shall be zero if a/b is: [IES-2001]

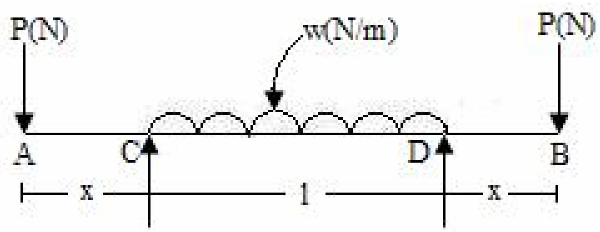
(a) 3/4 (b) 2/3 (c) 1/2 (d) 1/3

IES-3. Ans. (c)

Previous 20-Years IAS Questions

Overhanging Beam

IAS-1.



If the beam shown in the given figure is to have zero bending moment at its middle point, the overhang x should be: [IAS-2000]

(a) $wl^2/4P$ (b) $wl^2/6P$ (c) $wl^2/8P$ (d) $wl^2/12P$

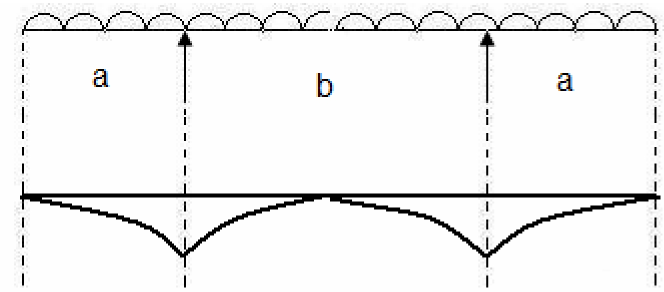
IAS-1. Ans. (c) $R_c = R_D = P + \dfrac{wl}{2}$

Bending moment at mid point (M) = $-\dfrac{wl}{2}\times\dfrac{l}{4} + R_D\times\dfrac{l}{2} - P\left(x+\dfrac{l}{2}\right) = 0 \; gives \; x = \dfrac{wl^2}{8P}$

IAS-2. A beam carrying a uniformly distributed load rests on two supports 'b' apart with equal overhangs 'a' at each end. The ratio b/a for zero bending moment at mid-span is: [IAS-1997]

(a) $\dfrac{1}{2}$ (b) 1 (c) $\dfrac{3}{2}$ (d) 2

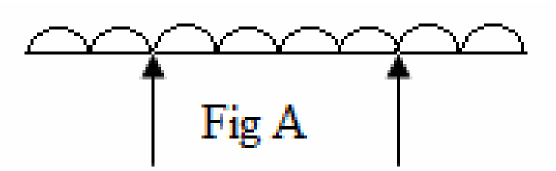
IAS-2. Ans. (d)

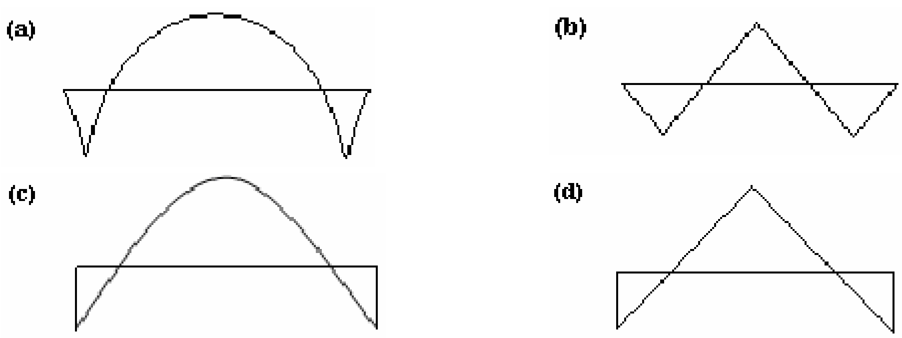


(i) By similarity in the B.M diagram a must be b/2

(ii) By formula $M = \frac{\omega}{2}\left[\frac{b^2}{4} - a^2\right] = 0$ gives a = b/2

IAS-3. A beam carries a uniformly distributed load and is supported with two equal overhangs as shown in figure 'A'. Which one of the following correctly shows the bending moment diagram of the beam? [IAS 1994]





IAS-3. Ans. (a)

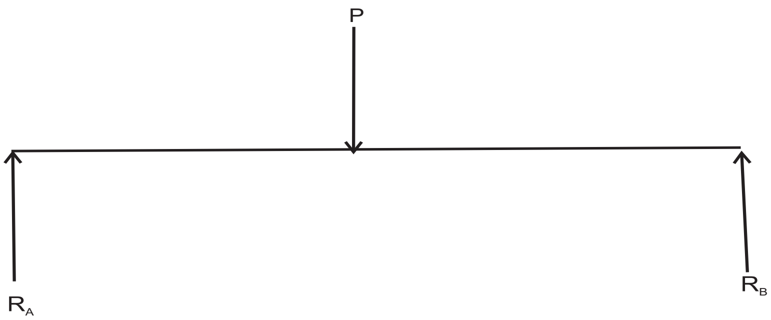
Previous Conventional Questions with Answers

Conventional Question IES-2006

Question: **What are statically determinate and in determinate beams? Illustrate each case through examples.**

Answer: Beams for which reaction forces and internal forces can be found out from static equilibrium equations alone are called statically determinate beam.

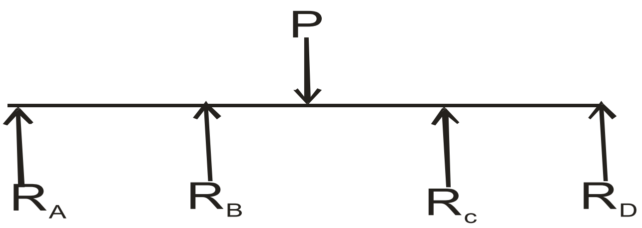
Example:

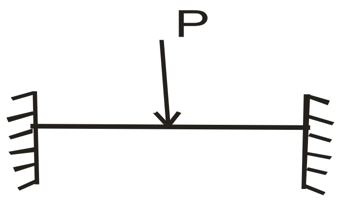


$\sum X_i = 0, \sum Y_i = 0$ and $\sum M_i = 0$ is sufficient to calculate R_A & R_B.

Beams for which reaction forces and internal forces **cannot** be found out from static equilibrium equations alone are called statically indeterminate beam. This type of beam requires deformation equation in addition to static equilibrium equations to solve for unknown forces.

Example:





9. Torsion

Theory at a Glance (for IES, GATE, PSU)

- In machinery, the general term ***"shaft"*** refers to a member, usually of circular cross-section, which supports gears, sprockets, wheels, rotors, etc., and which is subjected to torsion and to transverse or axial loads acting singly or in combination.
- An ***"axle"*** is a rotating/non-rotating member that supports wheels, pulleys,... and carries no torque.
- A ***"spindle"*** is a short shaft. Terms such as *lineshaft, headshaft, stub shaft, transmission shaft, countershaft,* and *flexible shaft* are names associated with special usage.

Torsion of circular shafts

1. Equation for shafts subjected to torsion "T"

$$\frac{\tau}{R} = \frac{T}{J} = \frac{G\theta}{L} \qquad \text{Torsion Equation}$$

Where J = Polar moment of inertia

τ = Shear stress induced due to torsion T.

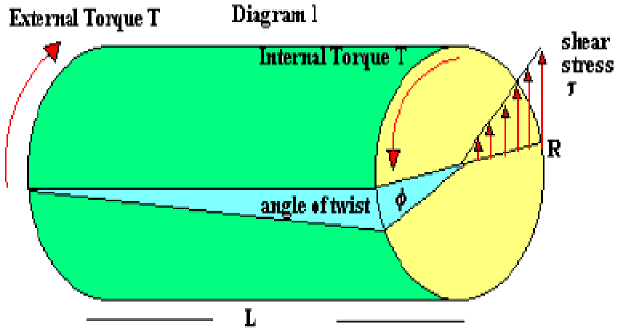
G = Modulus of rigidity

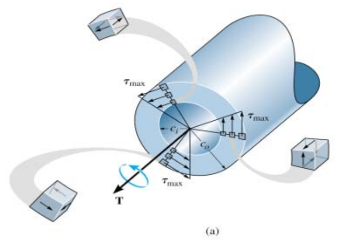
θ = Angular deflection of shaft

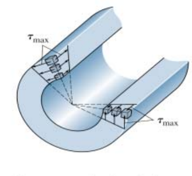
R, L = Shaft radius & length respectively

Assumptions

- The bar is acted upon by a pure torque.
- The section under consideration is remote from the point of application of the load and from a change in diameter.
- Adjacent cross sections originally plane and parallel remain plane and parallel after twisting, and any radial line remains straight.
- The material obeys Hooke's law
- Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle






Shear stress varies linearly along each radial line of the cross section.
(b)

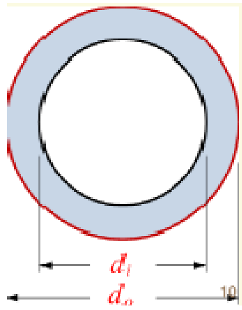
2. Polar moment of inertia

As stated above, the polar second moment of area, J is defined as

$$J = \int_0^R 2\pi r^3\, dr$$

For a solid shaft $$J = 2\pi \left[\frac{r^4}{4}\right]_0^R = \frac{2\pi R^4}{4} = \frac{\pi D^4}{32} \qquad (6)$$

For a hollow shaft of internal radius r:



$$J = \int_0^R 2\pi r^3\, dr = 2\pi \left[\frac{r^4}{4}\right]_r^R = \frac{\pi}{2}(R^4 - r^4) = \frac{\pi}{32}\left(D^4 - d^4\right) \qquad (7)$$

Where D is the external and d is the internal diameter.

- Solid shaft "J" = $\frac{\pi d^4}{32}$
- Hollow shaft, "J" = $\frac{\pi}{32}(d_o^4 - d_i^4)$

3. The polar section modulus

Z_p= J / c, where c = r = D/2

- For a solid circular cross-section, $Z_p = \pi D^3 / 16$
- For a hollow circular cross-section, $Z_p = \pi (D_o^4 - D_i^4) / (16 D_o)$
- Then, $\tau_{max} = T / Z_p$
- If design shears stress, τ_d is known, required polar section modulus can be calculated from: $Z_p = T / \tau_d$

4. Power Transmission (P)

- $P \text{ (in Watt)} = \frac{2\pi N T}{60}$

- P (in hp) $= \dfrac{2\pi NT}{4500}$ (1 hp = 75 Kgm/sec).

[Where N = rpm; T = Torque in N-m.]

5. Safe diameter of Shaft (d)

- Stiffness consideration

$$\frac{T}{J} = \frac{G\theta}{L}$$

- Shear Stress consideration

$$\frac{T}{J} = \frac{\tau}{R}$$

We take higher value of diameter of both cases above for overall safety if other parameters are given.

6. In twisting

- Solid shaft, $\tau_{\max} = \dfrac{16T}{\pi d^3}$
- Hollow shaft, $\tau_{\max} = \dfrac{16Td_o}{\pi(d_o^4 - d_i^4)}$
- Diameter of a shaft to have a maximum deflection "α" $\quad d = 4.9 \times \sqrt[4]{\dfrac{TL}{G\alpha}}$

[Where T in N-mm, L in mm, G in N/mm²]

7. Comparison of solid and hollow shaft

- A Hollow shaft will transmit a greater torque than a solid shaft of the same weight & same material because the average shear stress in the hollow shaft is smaller than the average shear stress in the solid shaft
- $\dfrac{(\tau_{\max})holloow\text{ shaft}}{(\tau_{\max})solid\text{ shaft}} = \dfrac{16}{15}$ $\quad \left[\begin{array}{l}\text{If solid shaft dia} = D \\ \text{Hollow shaft, } d_o = D, d_i = \dfrac{D}{2}\end{array}\right]$
- Strength comparison (same weight, material, length and $\tau_{\max}$)

$$\frac{T_h}{T_s} = \frac{n^2+1}{n\sqrt{n^2-1}} \quad \text{Where, } n = \frac{\text{External diameter of hollow shaft}}{\text{Internal diameter of hollow shaft}}$$ **[ONGC-2005]**

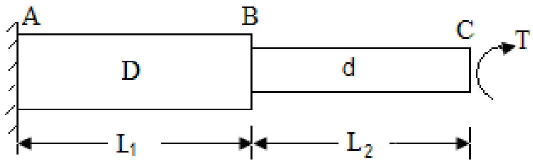
- Weight comparison (same Torque, material, length and $\tau_{\max}$)

$$\frac{W_h}{W_s} = \frac{\left(n^2-1\right)n^{2/3}}{\left(n^4-1\right)^{2/3}} \quad \text{Where, } n = \frac{\text{External diameter of hollow shaft}}{\text{Internal diameter of hollow shaft}}$$ **[WBPSC-2003]**

- Strain energy comparison (same weight, material, length and $\tau_{\max}$)

$$\frac{U_h}{U_s} = \frac{n^2+1}{n^2} = 1 + \frac{1}{n^2}$$

8. Shaft in series

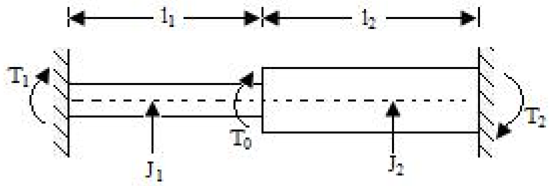


$\theta = \theta_1 + \theta_2$

Torque (T) is same in all section

Electrical analogy gives torque(T) = Current (I)

9. Shaft in parallel



$\theta_1 = \theta_2$ and $T = T_1 + T_2$

Electrical analogy gives torque(T) = Current (I)

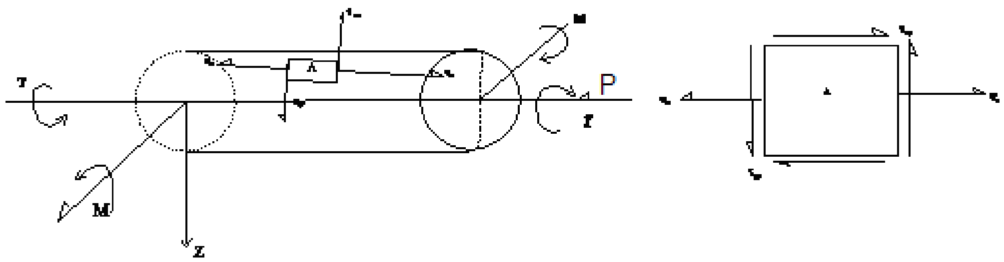
10. Combined Bending and Torsion

- In most practical transmission situations shafts which carry torque are also subjected **to** bending, if only by virtue of the self-weight of the gears they carry. Many other practical applications occur where bending and torsion arise simultaneously so that this type of loading represents one of the major sources of complex stress situations.

- In the case of shafts, bending gives rise to tensile stress on one surface and compressive stress on the opposite surface while torsion gives rise to pure shear throughout the shaft.

- For shafts subjected to the simultaneous application of a bending moment *M* and torque *T* the *principal stresses* set up in the shaft can be shown to be equal to those produced by an *equivalent bending moment,* of a certain value M_e acting alone.

- **Figure**



- Maximum direct stress (σ_x) & Shear stress ((τ_{xy}) in element A

$$\sigma_x = \frac{32M}{\pi d^3} + \frac{P}{A}$$

$$\tau_{xy} = \frac{16T}{\pi d^3}$$

- Principal normal stresses ($\sigma_{1,2}$) & Maximum shearing stress ($\tau_{\max}$)

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \pm\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + (\tau_{xy})^2}$$

- Maximum Principal Stress ($\sigma_{\max}$) & Maximum shear stress ($\tau_{\max}$)

$$\sigma_{\max} = \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

$$\tau_{\max} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2}$$

- Location of Principal plane (θ)

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{T}{M}\right)$$

- Equivalent bending moment (M_e) & Equivalent torsion (T_e).

$$M_e = \left[\frac{M + \sqrt{M^2 + T^2}}{2}\right]$$

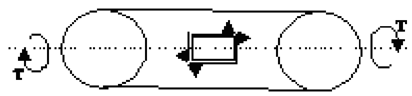
$$T_e = \sqrt{M^2 + T^2}$$

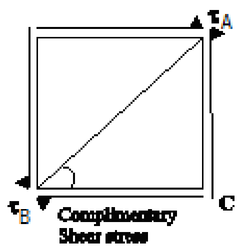
- **Important Note**
  - Uses of the formulas are limited to cases in which both M & T are known. Under any other condition Mohr's circle is used.

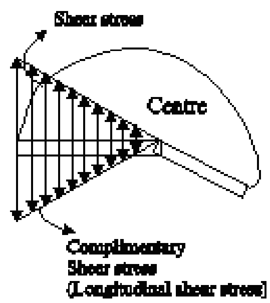
- Safe diameter of shaft (d) on the basis of an allowable working stress.
  - σ_w in tension , d = $\sqrt[3]{\dfrac{32M_e}{\pi\sigma_w}}$
  - τ_w in shear , d= $\sqrt[3]{\dfrac{16T_e}{\pi\tau_w}}$

11. Shaft subjected to twisting moment only

- **Figure**







- Normal force (F_n) & Tangential for (F_t) on inclined plane AB

$$F_n = -\tau \times [BC \sin\theta + AC \cos\theta]$$
$$F_t = \tau \times [BC \cos\theta - AC \sin\theta]$$

- Normal stress (σ_n) & Tangential stress (shear stress) (σ_t) on inclined plane AB.

$$\sigma_n = -\tau \sin 2\theta$$
$$\sigma_t = \tau cos 2\theta$$

- Maximum normal & shear stress on AB

θ	$(\sigma_n)_{max}$	τ_{max}
0	0	$+\tau$
45°	$-\tau$	0
90	0	$-\tau$
135	$+\tau$	0

- **Important Note**
  - Principal stresses at a point on the surface of the shaft = $+\tau$, $-\tau$, 0

    i.e $\sigma_{1,2} = \pm\ \tau \sin 2\theta$

  - Principal strains

    $$\in_1 = \frac{\tau}{E}(1+\mu); \quad \in_2 = -\frac{\tau}{E}(1+\mu); \quad \in_3 = 0$$

  - Volumetric strain,

    $$\in_v = \in_1 + \in_2 + \in_3 = 0$$

  - No change in volume for a shaft subjected to pure torque.

12. Torsional Stresses in Non-Circular Cross-section Members

- There are some applications in machinery for non-circular cross-section members and shafts where a regular polygonal cross-section is useful in transmitting torque to a gear or pulley that can have an axial change in position. Because no key or keyway is needed, the possibility of a lost key is avoided.

- Saint Venant (1855) showed that τ_{max} in a rectangular $b \times c$ section bar occurs in the middle of the longest side b and is of magnitude formula

$$\boldsymbol{\tau_{max} = \frac{T}{\alpha bc^2} = \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right)}$$

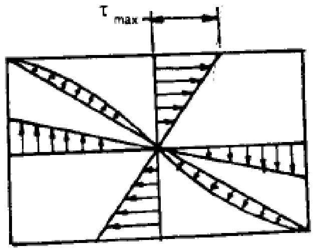
Where b is the longer side and α factor that is function of the ratio b/c.

The angle of twist is given by

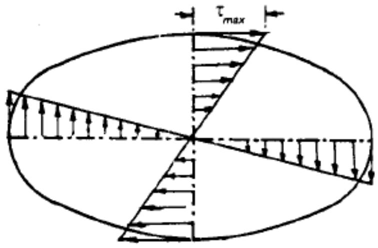
$$\theta = \frac{Tl}{\beta bc^3 G}$$

Where β is a function of the ratio b/c

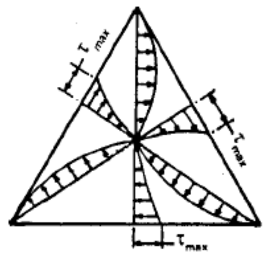
Shear stress distribution in different cross-section



Rectangular c/s



Elliptical c/s



Triangular c/s

13. Torsion of thin walled tube

- For a thin walled tube

  Shear stress, $\tau = \dfrac{T}{2A_0 t}$

  Angle of twist, $\phi = \dfrac{\tau s L}{2A_O G}$

  [Where S = length of mean centre line, A_O = Area enclosed by mean centre line]

- **Special Cases**
  - For circular c/s

    $J = 2\pi r^3 t; \qquad A_o = \pi r^2; \qquad S = 2\pi r$

    [r = radius of mean Centre line and t = wall thickness]

    $$\therefore \ \tau = \frac{T}{2\pi r^2 t} = \frac{T.r}{J} = \frac{T}{2A_o t}$$

    $$\varphi = \frac{TL}{GJ} = \frac{\tau L}{A_o JG} = \frac{TL}{2\pi r^3 tG}$$

  - For square c/s of length of each side 'b' and thickness 't'

    $A_0 = b^2$

    $S = 4b$

  - For elliptical c/s 'a' and 'b' are the half axis lengths.

    $A_0 = \pi ab$

    $$S \approx \pi\left[\frac{3}{2}(a+b) - \sqrt{ab}\right]$$

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Torsion Equation

GATE-1. A solid circular shaft of 60 mm diameter transmits a torque of 1600 N.m. The value of maximum shear stress developed is: [GATE-2004]
(a) 37.72 MPa (b) 47.72 MPa (c) 57.72 MPa (d) 67.72 MPa

GATE-1. Ans. (a) $\tau = \frac{16T}{\pi d^3}$

GATE-2. Maximum shear stress developed on the surface of a solid circular shaft under pure torsion is 240 MPa. If the shaft diameter is doubled then the maximum shear stress developed corresponding to the same torque will be: [GATE-2003]
(a) 120 MPa (b) 60 MPa (c) 30 MPa (d) 15 MPa

GATE-2. Ans. (c) $\tau = \frac{16T}{\pi d^3}$, $240 = \frac{16T}{\pi d^3}$ if diameter doubled $d' = 2d$, then $\tau' = \frac{16T}{\pi (2d)^3} = \frac{240}{8} = 30\text{MPa}$

GATE-3. A steel shaft 'A' of diameter 'd' and length 'l' is subjected to a torque 'T' Another shaft 'B' made of aluminium of the same diameter 'd' and length 0.5l is also subjected to the same torque 'T'. The shear modulus of steel is 2.5 times the shear modulus of aluminium. The shear stress in the steel shaft is 100 MPa. The shear stress in the aluminium shaft, in MPa, is: [GATE-2000]
(a) 40 (b) 50 (c) 100 (d) 250

GATE-3. Ans. (c) $\tau = \frac{16T}{\pi d^3}$ as T & d both are same τ is same

GATE-4. For a circular shaft of diameter d subjected to torque T, the maximum value of the shear stress is: [GATE-2006]
(a) $\frac{64T}{\pi d^3}$ (b) $\frac{32T}{\pi d^3}$ (c) $\frac{16T}{\pi d^3}$ (d) $\frac{8T}{\pi d^3}$

GATE-4. Ans. (c)

Power Transmitted by Shaft

GATE-5. The diameter of shaft A is twice the diameter or shaft B and both are made of the same material. Assuming both the shafts to rotate at the same speed, the maximum power transmitted by B is: [IES-2001; GATE-1994]
(a) The same as that of A (b) Half of A (c) 1/8th of A (d) 1/4th of A

GATE-5. Ans. (c) Power, $P = T \times \frac{2\pi N}{60}$ and $\tau = \frac{16T}{\pi d^3}$ or $T = \frac{\tau \pi d^3}{16}$

or $P = \frac{\tau \pi d^3}{16} \times \frac{2\pi N}{60}$ or $P \ \alpha \ d^3$

Combined Bending and Torsion

GATE-6. A solid shaft can resist a bending moment of 3.0 kNm and a twisting moment of 4.0 kNm together, then the maximum torque that can be applied is: [GATE-1996]
(a) 7.0 kNm (b) 3.5 kNm (c)4.5 kNm (d) 5.0 kNm

GATE-6. Ans. (d) Equivalent torque $(T_e) = \sqrt{M^2 + T^2} = \sqrt{3^2 + 4^2} = 5\text{kNm}$

Comparison of Solid and Hollow Shafts

GATE-7. The outside diameter of a hollow shaft is twice its inside diameter. The ratio of its torque carrying capacity to that of a solid shaft of the same material and the same outside diameter is: [GATE-1993; IES-2001]

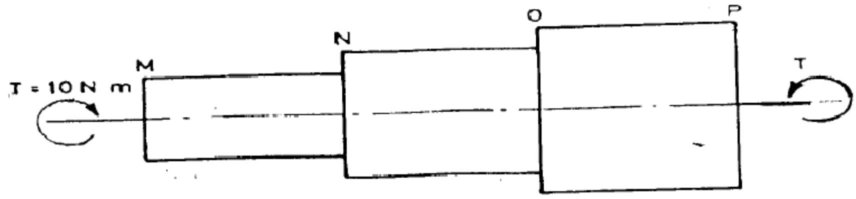
(a) $\frac{15}{16}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{16}$

GATE-7. Ans. (a) $\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$ or $T = \frac{\tau J}{R}$ if τ is const. $T \ \alpha \ J$

$$\frac{T_h}{T} = \frac{J_h}{J} = \frac{\frac{\pi}{32}\left[D^4 - \left(\frac{D}{2}\right)^4\right]}{\frac{\pi}{32}D^4} = \frac{15}{16}$$

Shafts in Series

GATE-8. A torque of 10 Nm is transmitted through a stepped shaft as shown in figure. The torsional stiffness of individual sections of lengths MN, NO and OP are 20 Nm/rad, 30 Nm/rad and 60 Nm/rad respectively. The angular deflection between the ends M and P of the shaft is: [GATE-2004]



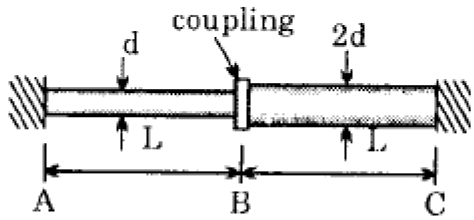
(a) 0.5 rad (b) 1.0 rad (c) 5.0 rad (d) 10.0 rad

GATE-8. Ans. (b) We know that $\theta = \frac{TL}{GJ}$ or $T = k.\theta$ [let k = tortional stiffness]

$$\therefore \theta = \theta_{MN} + \theta_{NO} + \theta_{OP} = \frac{T_{MN}}{k_{MN}} + \frac{T_{NO}}{k_{NO}} + \frac{T_{OP}}{k_{OP}} = \frac{10}{20} + \frac{10}{30} + \frac{10}{60} = 1.0 \text{ rad}$$

Shafts in Parallel

GATE-9. The two shafts AB and BC, of equal length and diameters d and 2d, are made of the same material. They are joined at B through a shaft coupling, while the ends A and C are built-in (cantilevered). A twisting moment T is applied to the coupling. If T_A and T_C represent the twisting moments at the ends A and C, respectively, then [GATE-2005]



(a) $T_C = T_A$ (b) $T_C = 8\ T_A$ (c) $T_C = 16\ T_A$ (d) $T_A = 16\ T_C$

GATE-9. Ans. (c) $\theta_{AB} = \theta_{BC}$ or $\frac{T_A L_A}{G_A J_A} = \frac{T_C L_C}{G_C J_C}$ or $\frac{T_A}{\frac{\pi d^4}{32}} = \frac{T_C}{\frac{\pi (2d)^4}{32}}$ or $T_A = \frac{T_C}{16}$

Previous 20-Years IES Questions

Torsion Equation

IES-1. Consider the following statements: [IES- 2008]
Maximum shear stress induced in a power transmitting shaft is:
1. Directly proportional to torque being transmitted.
2. Inversely proportional to the cube of its diameter.

3. Directly proportional to its polar moment of inertia.
Which of the statements given above are correct?
(a) 1, 2 and 3 (b) 1 and 3 only (c) 2 and 3 only (d) 1 and 2 only

IES-1. Ans. (d) $\tau = \frac{T \times r}{J} = \frac{16T}{\pi d^3}$

IES-2. A solid shaft transmits a torque T. The allowable shearing stress is τ. What is the diameter of the shaft? [IES-2008]
(a) $\sqrt[3]{\frac{16T}{\pi\tau}}$ (b) $\sqrt[3]{\frac{32T}{\pi\tau}}$ (c) $\sqrt[3]{\frac{16T}{\tau}}$ (d) $\sqrt[3]{\frac{T}{\tau}}$

IES-2. Ans. (a)

IES-3. Maximum shear stress developed on the surface of a solid circular shaft under pure torsion is 240 MPa. If the shaft diameter is doubled, then what is the maximum shear stress developed corresponding to the same torque? [IES-2009]
(a) 120 MPa (b) 60 MPa (c) 30 MPa (d) 15 MPa

IES-3. Ans. (c) Maximum shear stress = $\frac{16T}{\pi d^3}$ = 240 MPa = τ

Maximum shear stress developed when diameter is doubled

$$= \frac{16\tau}{\pi(2d)^3} = \frac{1}{8}\left(\frac{16T}{\pi d^3}\right) = \frac{\tau}{8} = \frac{240}{8} = 30\,\text{MPa}$$

IES-4. The diameter of a shaft is increased from 30 mm to 60 mm, all other conditions remaining unchanged. How many times is its torque carrying capacity increased? [IES-1995; 2004]
(a) 2 times (b) 4 times (c) 8 times (d) 16 times

IES-4. Ans. (c) $\tau = \frac{16T}{\pi d^3}$ or $T = \frac{\tau\pi d^3}{16}$ for same material $\tau = \text{const.}$

$$\therefore T \alpha\ d^3 \quad \text{or} \quad \frac{T_2}{T_1} = \left(\frac{d_2}{d_1}\right)^3 = \left(\frac{60}{30}\right)^3 = 8$$

IES-5. A circular shaft subjected to twisting moment results in maximum shear stress of 60 MPa. Then the maximum compressive stress in the material is: [IES-2003]
(a) 30 MPa (b) 60 MPa (c) 90 MPa (d) 120 MPa

IES-5. Ans. (b)

IES-6. Angle of twist of a shaft of diameter 'd' is inversely proportional to [IES-2000]
(a) d (b) d^2 (c) d^3 (d) d^4

IES-6. Ans. (d)

IES-7. A solid circular shaft is subjected to pure torsion. The ratio of maximum shear to maximum normal stress at any point would be: [IES-1999]
(a) 1 : 1 (b) 1: 2 (c) 2: 1 (d) 2: 3

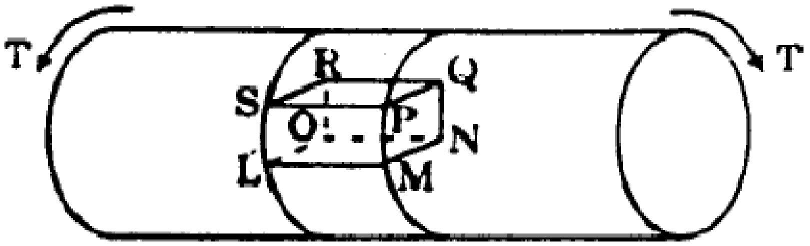
IES-7. Ans. (a) Shear stress $= \frac{16T}{\pi d^3}$ and normal stress $= \frac{32T}{\pi d^3}$

$\therefore$ Ratio of shear stress and normal stress = 1: 2

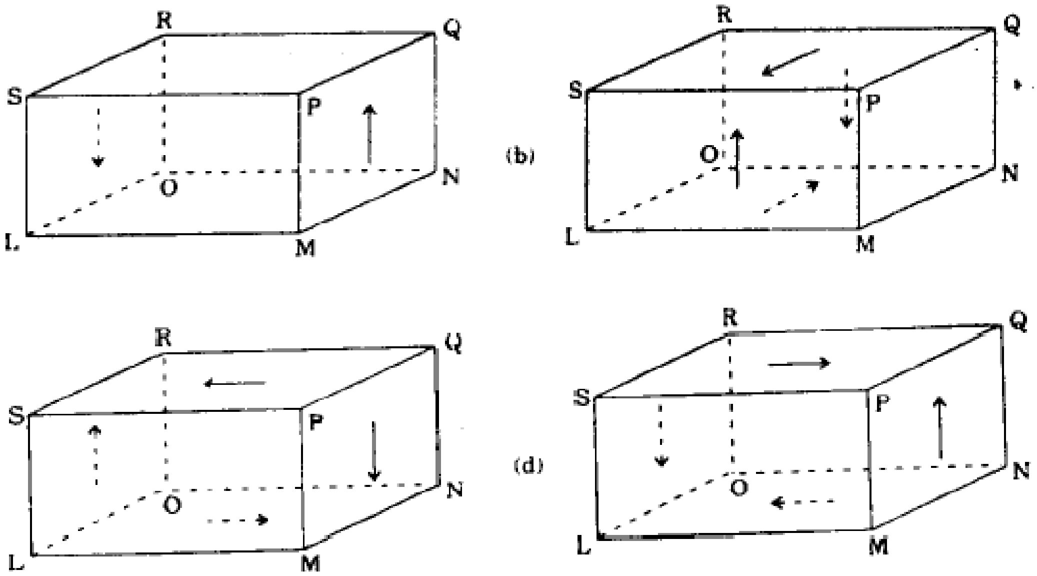
IES-8. Assertion (A): In a composite shaft having two concentric shafts of different materials, the torque shared by each shaft is directly proportional to its polar moment of inertia. [IES-1999]
Reason (R): In a composite shaft having concentric shafts of different materials, the angle of twist for each shaft depends upon its polar moment of inertia.
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-8. Ans. (c)

IES-9. A shaft is subjected to torsion as shown. [IES-2002]

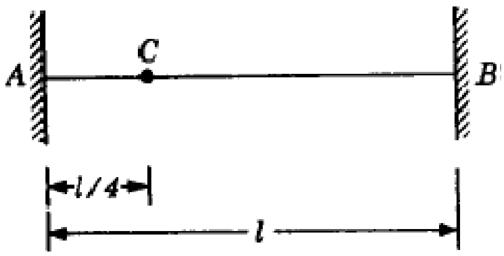


Which of the following figures represents the shear stress on the element LMNOPQRS ?



IES-9. Ans. (d)

IES-10. A round shaft of diameter 'd' and length 'l' fixed at both ends 'A' and 'B' is subjected to a twisting moment 'T' at 'C', at a distance of 1/4 from A (see figure). The torsional stresses in the parts AC and CB will be:



(a) Equal
(b) In the ratio 1:3
(c) In the ratio 3 :1
(d) Indeterminate

[IES-1997]

IES-10. Ans. (c) $\frac{T}{J}=\frac{\tau}{R}=\frac{G\theta}{L}$ or $\tau=\frac{GR\theta}{L}$ $\therefore \tau \propto \frac{1}{L}$

Hollow Circular Shafts

IES-11. One-half length of 50 mm diameter steel rod is solid while the remaining half is hollow having a bore of 25 mm. The rod is subjected to equal and opposite torque at its ends. If the maximum shear stress in solid portion is τ or, the maximum shear stress in the hollow portion is: [IES-2003]

(a) $\frac{15}{16}\tau$ (b) τ (c) $\frac{4}{3}\tau$ (d) $\frac{16}{15}\tau$

IES-11. Ans. (d) $\frac{T}{J}=\frac{\tau}{r}$ or $T=\frac{\tau J}{r}$

$$or\ \frac{\tau J_s}{r_s}=\frac{\tau_h J_h}{r_h};\ \left[r_s=r_h=\frac{D}{2}\right]$$

$$or\ \tau_h = \tau \times \frac{J_s}{J_h} = \tau \times \frac{\frac{\pi}{32}D^4}{\frac{\pi}{32}\left(D^4 - d^4\right)} = \tau \times \frac{1}{\left[1-\left(\frac{d}{D}\right)^4\right]} = \tau \times \frac{1}{\left[1-\left(\frac{25}{50}\right)^4\right]} = \tau\left(\frac{16}{15}\right)$$

Power Transmitted by Shaft

IES-12. In power transmission shafts, if the polar moment of inertia of a shaft is doubled, then what is the torque required to produce the same angle of twist? [IES-2006]

(a) 1/4 of the original value (b) 1/2 of the original value
(c) Same as the original value (d) Double the original value

IES-12. Ans. (d)

$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$ or $Q = \frac{T.L}{G.J}$ if θ is const. $T\ \alpha\ J$ if J is doubled then T is also doubled.

IES-13. While transmitting the same power by a shaft, if its speed is doubled, what should be its new diameter if the maximum shear stress induced in the shaft remains same? [IES-2006]

(a) $\frac{1}{2}$ of the original diameter (b) $\frac{1}{\sqrt{2}}$ of the original diameter

(c) $\sqrt{2}$ of the original diameter (d) $\frac{1}{(2)^{1/3}}$ of the original diameter

IES-13. Ans. (d) Power $(P) = \text{torque}(T) \times \text{angular speed}(\omega)$

if P is const. $T\alpha\frac{1}{\omega}$ if $\frac{T'}{T} = \frac{\omega}{\omega'} = \frac{1}{2}$ or $T' = (T/2)$

$\sigma = \frac{16T}{\pi d^3} = \frac{16(T/2)}{\pi(d')^3}$ or $\left(\frac{d'}{d}\right) = \frac{1}{\sqrt[3]{2}}$

IES-14. For a power transmission shaft transmitting power P at N rpm, its diameter is proportional to: [IES-2005]

(a) $\left(\frac{P}{N}\right)^{1/3}$ (b) $\left(\frac{P}{N}\right)^{1/2}$ (c) $\left(\frac{P}{N}\right)^{2/3}$ (d) $\left(\frac{P}{N}\right)$

IES-14. Ans. (a) Power, $P = T \times \frac{2\pi N}{60}$ and $\tau = \frac{16T}{\pi d^3}$ or $T = \frac{\tau\pi d^3}{16}$

or $P = \frac{\tau\pi d^3}{16} \times \frac{2\pi N}{60}$ or $d^3 = \frac{480\ P}{\pi^2 J\ N}$ or $d\ \alpha\ \left(\frac{P}{N}\right)^{1/3}$

IES-15. A shaft can safely transmit 90 kW while rotating at a given speed. If this shaft is replaced by a shaft of diameter double of the previous one and rotated at half the speed of the previous, the power that can be transmitted by the new shaft is: [IES-2002]

(a) 90 kW (b) 180 kW (c) 360 kW (d) 720 kW

IES-15. Ans. (c)

IES-16. The diameter of shaft A is twice the diameter or shaft B and both are made of the same material. Assuming both the shafts to rotate at the same speed, the maximum power transmitted by B is: [IES-2001; GATE-1994]

(a) The same as that of A (b) Half of A (c) 1/8th of A (d) 1/4th of A

IES-16. Ans. (c) Power, $P = T \times \frac{2\pi N}{60}$ and $\tau = \frac{16T}{\pi d^3}$ or $T = \frac{\tau\pi d^3}{16}$

or $P = \frac{\tau\pi d^3}{16} \times \frac{2\pi N}{60}$ or $P\ \alpha\ d^3$

IES-17. When a shaft transmits power through gears, the shaft experiences [IES-1997]
(a) Torsional stresses alone
(b) Bending stresses alone
(c) Constant bending and varying torsional stresses
(d) Varying bending and constant torsional stresses

IES-17. Ans. (d)

Combined Bending and Torsion

IES-18. The equivalent bending moment under combined action of bending moment M and torque T is: [IES-1996; 2008; IAS-1996]

(a) $\sqrt{M^2+T^2}$ (b) $\frac{1}{2}\left[M+\sqrt{M^2+T^2}\right]$

(c) $\frac{1}{2}\left[M+T\right]$ (d) $\frac{1}{4}\left[\sqrt{M^2+T^2}\right]$

IES-18. Ans. (b)

IES-19. A solid circular shaft is subjected to a bending moment M and twisting moment T. What is the equivalent twisting moment T_e which will produce the same maximum shear stress as the above combination? [IES-1992; 2007]

(a) $M^2 + T^2$ (b) M + T (c) $\sqrt{M^2+T^2}$ (d) M – T

IES-19. Ans. (c) $T_e = \sqrt{M^2 + T^2}$

IES-20. A shaft is subjected to fluctuating loads for which the normal torque (T) and bending moment (M) are 1000 N-m and 500 N-m respectively. If the combined shock and fatigue factor for bending is 1.5 and combined shock and fatigue factor for torsion is 2, then the equivalent twisting moment for the shaft is: [IES-1994]

(a) 2000N-m (b) 2050N-m (c) 2100N-m (d) 2136 N-m

IES-20. Ans. (d) $T_{eq} = \sqrt{(1.5\times500)^2 + (2\times1000)^2} = 2136\,\text{Nm}$

IES-21. A member is subjected to the combined action of bending moment 400 Nm and torque 300 Nm. What respectively are the equivalent bending moment and equivalent torque? [IES-1994; 2004]

(a) 450 Nm and 500 Nm (b) 900 Nm and 350 Nm
(c) 900 Nm and 500 Nm (d) 400 Nm and 500 Nm

IES-21. Ans. (a) Equivalent Bending Moment $(M_e) = \dfrac{M+\sqrt{M^2+T^2}}{2} = \dfrac{400+\sqrt{400^2+300^2}}{2} = 450\text{N.m}$

Equivalent torque $(T_e) = \sqrt{M^2+T^2} = \sqrt{400^2+300^2} = 500\text{N.m}$

IES-22. A shaft was initially subjected to bending moment and then was subjected to torsion. If the magnitude of bending moment is found to be the same as that of the torque, then the ratio of maximum bending stress to shear stress would be: [IES-1993]

(a) 0.25 (b) 0.50 (c) 2.0 (d) 4.0

IES-22. Ans. (c) Use equivalent bending moment formula,

1st case: Equivalent bending moment (M_e) = M

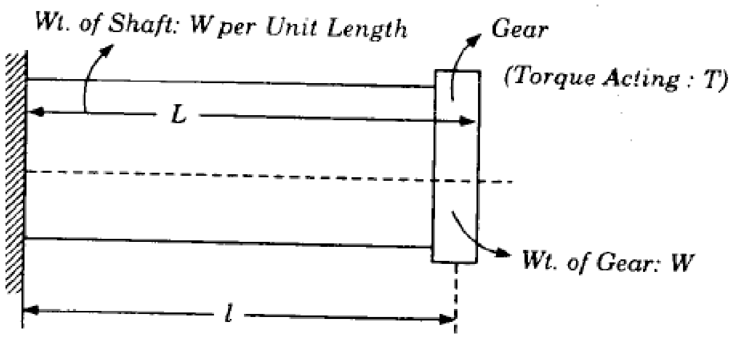
2nd case: Equivalent bending moment $(M_e) = \dfrac{0+\sqrt{0^2+T^2}}{2} = \dfrac{T}{2}$

IES-23. A shaft is subjected to simultaneous action of a torque T, bending moment M and an axial thrust F. Which one of the following statements is correct for this situation? [IES-2004]

(a) One extreme end of the vertical diametral fibre is subjected to maximum compressive stress only
(b) The opposite extreme end of the vertical diametral fibre is subjected to tensile/compressive stress only
(c) Every point on the surface of the shaft is subjected to maximum shear stress only
(d) Axial longitudinal fibre of the shaft is subjected to compressive stress only

IES-23. Ans. (a)

IES-24. For obtaining the maximum shear stress induced in the shaft shown in the given figure, the torque should be equal to



(a) T (b) $Wl + T$

(c) $\left[(Wl)^2 + \left(\frac{wL}{2}\right)^2\right]^{\frac{1}{2}}$

(d) $\left[\left\{Wl + \frac{wL^2}{2}\right\}^2 + T^2\right]^{\frac{1}{2}}$

[IES-1999]

IES-24. Ans. (d) Bending Moment, M = $Wl + \frac{wL^2}{2}$

IES-25. Bending moment M and torque is applied on a solid circular shaft. If the maximum bending stress equals to maximum shear stress developed, them M is equal to: [IES-1992]

(a) $\frac{T}{2}$ (b) T (c) $2T$ (d) $4T$

IES-25. Ans. (a) $\sigma = \frac{32 \times \mathrm{M}}{\pi \mathrm{d}^3}$ and $\tau = \frac{16\mathrm{T}}{\pi \mathrm{d}^3}$

IES-26. A circular shaft is subjected to the combined action of bending, twisting and direct axial loading. The maximum bending stress σ, maximum shearing force $\sqrt{3}\sigma$ and a uniform axial stress σ(compressive) are produced. The maximum compressive normal stress produced in the shaft will be: [IES-1998]

(a) 3 σ (b) 2 σ (c) σ (d) Zero

IES-26. Ans. (a) Maximum normal stress = bending stress σ + axial stress (σ) = 2 σ

We have to take maximum bending stress σ is (compressive)

The maximum compressive normal stress = $\frac{\sigma_b}{2} - \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau_{xy}^2}$

$$= \frac{-2\sigma}{2} - \sqrt{\left(\frac{-2\sigma}{2}\right)^2 + \left(\sqrt{3}\sigma\right)^2} = -3\sigma$$

IES-27. Which one of the following statements is correct? Shafts used in heavy duty speed reducers are generally subjected to: [IES-2004]

(a) Bending stress only

(b) Shearing stress only
(c) Combined bending and shearing stresses
(d) Bending, shearing and axial thrust simultaneously

IES-27. Ans. (c)

Comparison of Solid and Hollow Shafts

IES-28. The ratio of torque carrying capacity of a solid shaft to that of a hollow shaft is given by: [IES-2008]

(a) $(1-K^4)$ (b) $(1-K^4)^{-1}$ (c) K^4 (d) $1/K^4$

Where $K = \frac{D_i}{D_o}$; D_i = Inside diameter of hollow shaft and D_o = Outside diameter of hollow shaft. Shaft material is the same.

IES-28. Ans. (b) τ should be same for both hollow and solid shaft

$$\frac{T_s}{\frac{\pi}{32}D_o^4} = \frac{T_h}{\frac{\pi}{32}\left(D_o^4 - D_i^4\right)} \Rightarrow \frac{T_s}{T_h} = \frac{D_o^4}{D_o^4 - D_i^4} \Rightarrow \frac{T_s}{T_h} = \left(1-\left(\frac{D_i}{D_o}\right)^4\right)^{-1}$$

$$\therefore \frac{T_s}{T_h} \left(1-k^4\right)^{-1}$$

IES-29. A hollow shaft of outer dia 40 mm and inner dia of 20 mm is to be replaced by a solid shaft to transmit the same torque at the same maximum stress. What should be the diameter of the solid shaft? [IES 2007]

(a) 30 mm (b) 35 mm (c) $10\times(60)^{1/3}$ mm (d) $10\times(20)^{1/3}$ mm

IES-29. Ans. (c) Section modules will be same

$$\frac{J_H}{R_H} = \frac{J_s}{R_s} \text{ or } \frac{\frac{\pi}{64}(40^4 - 20^4)}{\frac{40}{2}} = \frac{\pi}{64} \times \frac{d^4}{d/2}$$

or, $d^3 = (10)^3 \times 60$ or $d = 10\sqrt[3]{60}$ mm

IES-30. The diameter of a solid shaft is D. The inside and outside diameters of a hollow shaft of same material and length are $\frac{D}{\sqrt{3}}$ and $\frac{2D}{\sqrt{3}}$ respectively. What is the ratio of the weight of the hollow shaft to that of the solid shaft? [IES 2007]

(a) 1:1 (b) $1:\sqrt{3}$ (c) 1:2 (d) 1:3

IES-30. Ans. (a) $$\frac{W_H}{W_S} = \frac{\frac{\pi}{4}\left(\frac{4D^2}{3} - \frac{D^2}{3}\right)\times L\times\rho\times g}{\frac{\pi}{4}D^2\times L\times\rho\times g} = 1$$

IES-31. What is the maximum torque transmitted by a hollow shaft of external radius R and internal radius r? [IES-2006]

(a) $\frac{\pi}{16}\left(R^3 - r^3\right)f_s$ (b) $\frac{\pi}{2R}\left(R^4 - r^4\right)f_s$ (c) $\frac{\pi}{8R}\left(R^4 - r^4\right)f_s$ (d) $\frac{\pi}{32}\left(\frac{R^4 - r^4}{R}\right)f_s$

(f_s = maximum shear stress in the shaft material)

IES-31. Ans. (b) $\frac{T}{J} = \frac{f_s}{R}$ or $T = \frac{J}{R}\times f_s = \frac{\frac{\pi}{2}\left(R^4 - r^4\right)}{R}\times f_s = \frac{\pi}{2R}\left(R^4 - r^4\right).f_s.$

IES-32. A hollow shaft of the same cross-sectional area and material as that of a solid shaft transmits: [IES-2005]

(a) Same torque (b) Lesser torque
(c) More torque (d) Cannot be predicted without more data

IES-32. Ans. (c) $\frac{T_H}{T_S} = \frac{n^2+1}{n\sqrt{n^2-1}}$, Where $n = \frac{D_H}{d_H}$

IES-33. The outside diameter of a hollow shaft is twice its inside diameter. The ratio of its torque carrying capacity to that of a solid shaft of the same material and the same outside diameter is: [GATE-1993; IES-2001]

(a) $\frac{15}{16}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{16}$

IES-33. Ans. (a) $\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$ or $T = \frac{\tau J}{R}$ if τ is const. $T \;\alpha\; J$

$$\frac{T_h}{T} = \frac{J_h}{J} = \frac{\frac{\pi}{32}\left[D^4 - \left(\frac{D}{2}\right)^4\right]}{\frac{\pi}{32}D^4} = \frac{15}{16}$$

IES-34. Two hollow shafts of the same material have the same length and outside diameter. Shaft 1 has internal diameter equal to one-third of the outer diameter and shaft 2 has internal diameter equal to half of the outer diameter. If both the shafts are subjected to the same torque, the ratio of their twists θ_1/θ_2 will be equal to: [IES-1998]

(a) 16/81 (b) 8/27 (c) 19/27 (d) 243/256

IES-34. Ans. (d) $Q \infty \frac{1}{J} \quad \therefore \frac{Q_1}{Q_2} = \frac{d_1^4 - \left(d_1/2\right)^4}{d_1^4 - \left(d_1/3\right)^4} = \frac{243}{256}$

IES-35. Maximum shear stress in a solid shaft of diameter D and length L twisted through an angle θ is τ. A hollow shaft of same material and length having outside and inside diameters of D and D/2 respectively is also twisted through the same angle of twist θ. The value of maximum shear stress in the hollow shaft will be: [IES-1994; 1997]

(a) $\frac{16}{15}\tau$ (b) $\frac{8}{7}\tau$ (c) $\frac{4}{3}\tau$ (d) τ

IES-35. Ans. (d) $\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$ or $\tau = \frac{G.R.\theta}{L}$ if θ is const. $\tau \;\alpha\; R$ and outer diameter is same in both the cases.

Note: Required torque will be different.

IES-36. A solid shaft of diameter 'D' carries a twisting moment that develops maximum shear stress τ. If the shaft is replaced by a hollow one of outside diameter 'D' and inside diameter D/2, then the maximum shear stress will be: [IES-1994]

(a) 1.067 τ (b) 1.143 τ (c) 1.333 τ (d) 2 τ

IES-36. Ans. (a) $\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$ or $\tau = \frac{TR}{J}$ if T is const. $\tau \;\alpha\; \frac{1}{J}$

$$\frac{\tau_h}{\tau} = \frac{J}{J_h} = \frac{D^4}{D^4 - \left(\frac{D}{2}\right)^4} = \frac{16}{15} = 1.06666$$

IES-37. A solid shaft of diameter 100 mm, length 1000 mm is subjected to a twisting moment 'T' The maximum shear stress developed in the shaft is 60 N/mm². A hole of 50 mm diameter is now drilled throughout the length of the shaft. To

develop a maximum shear stress of 60 N/mm² in the hollow shaft, the torque 'T' must be reduced by: **[IES-1998]**

(a) T/4 (b) T/8 (c) T/12 (d)T/16

IES-37. Ans. (d) $\tau_s = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{T'32(d/2)}{d^4 - (d/2)^4}$ or $\frac{T'}{T} = \frac{15}{16}$

$\therefore$ Reduction $= \frac{1}{16}$

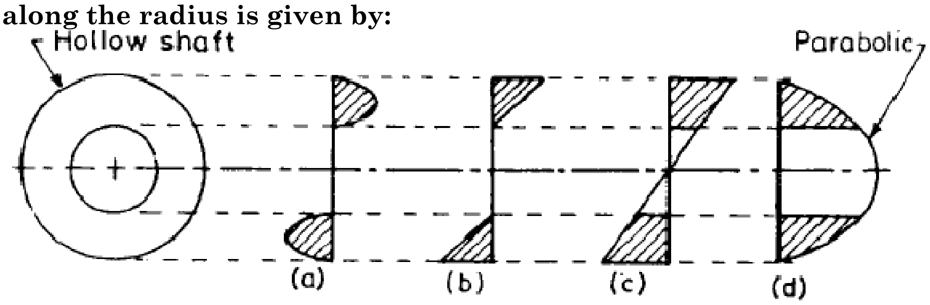
IES-38. Assertion (A): A hollow shaft will transmit a greater torque than a solid shaft of the same weight and same material. **[IES-1994]**

Reason (R): The average shear stress in the hollow shaft is smaller than the average shear stress in the solid shaft.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-38. Ans. (a)

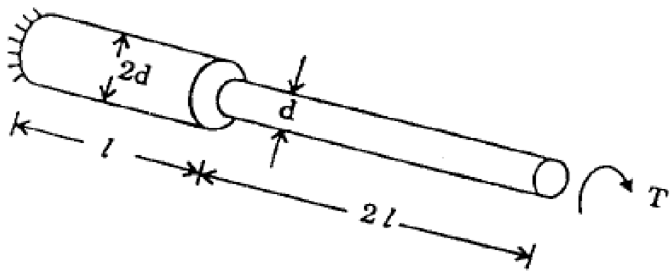
IES-39. A hollow shaft is subjected to torsion. The shear stress variation in the shaft along the radius is given by: **[IES-1996]**



IES-39. Ans. (c)

Shafts in Series

IES-40. What is the total angle of twist of the stepped shaft subject to torque T shown in figure given above?



(a) $\frac{16Tl}{\pi G d^4}$ (b) $\frac{38Tl}{\pi G d^4}$

(c) $\frac{64Tl}{\pi G d^4}$ (d) $\frac{66Tl}{\pi G d^4}$

[IES-2005]

IES-40. Ans. (d) $\theta = \theta_1 + \theta_2 = \frac{T \times 2l}{G.\frac{\pi d^4}{32}} + \frac{T \times l}{G \times \frac{\pi}{32} \times (2d)^4} = \frac{Tl}{Gd^4}[64 + 2] = \frac{66Tl}{Gd^4}$

Shafts in Parallel

IES-41. For the two shafts connected in parallel, find which statement is true?

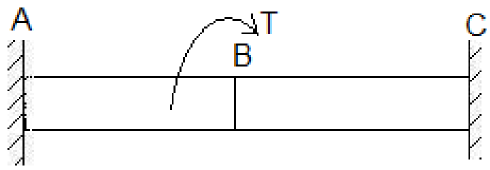
(a) Torque in each shaft is the same **[IES-1992]**
(b) Shear stress in each shaft is the same
(c) Angle of twist of each shaft is the same
(d) Torsional stiffness of each shaft is the same

IES-41. Ans. (c)

IES-42. **A circular section rod ABC is fixed at ends A and C. It is subjected to torque T at B. AB = BC = L and the polar moment of inertia of portions AB and BC are 2 J and J respectively. If G is the modulus of rigidity, what is the angle of twist at point B?** **[IES-2005]**

(a) $\frac{TL}{3GJ}$ (b) $\frac{TL}{2GJ}$ (c) $\frac{TL}{GJ}$ (d) $\frac{2TL}{GJ}$

IES-42. Ans. (a)



$\theta_{AB} = \theta_{BC}$

or $\frac{T_{AB}L}{G.2J} = \frac{T_{BC.}L}{G.J}$ or $T_{AB} = 2T_{BC}$

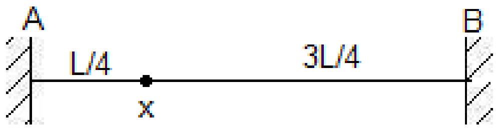
$T_{AB} + T_{BC} = T$ or $T_{BC} = T/3$

or $Q_B = Q_{AB} = \frac{T}{3}.\frac{L}{GJ} = \frac{TL}{3GJ}$

IES-43. **A solid circular rod AB of diameter D and length L is fixed at both ends. A torque T is applied at a section X such that AX = L/4 and BX = 3L/4. What is the maximum shear stress developed in the rod?** **[IES-2004]**

(a) $\frac{16T}{\pi D^3}$ (b) $\frac{12T}{\pi D^3}$ (c) $\frac{8T}{\pi D^3}$ (d) $\frac{4T}{\pi D^3}$

IES-43. Ans. (b)



$\theta_{AX} = \theta_{XB}$ & $T_A + T_B = T$

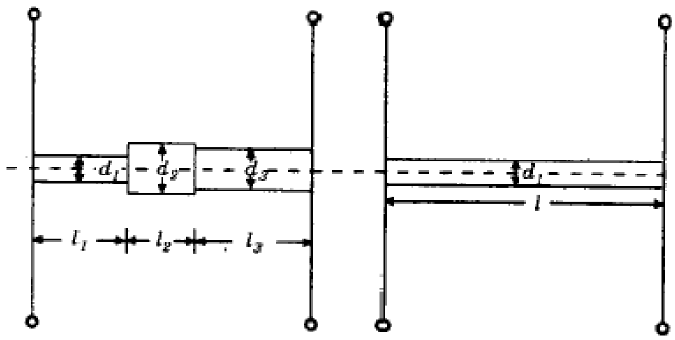
or $\frac{T_{A.}L/4}{GJ} = \frac{T_B \times \frac{3L}{4}}{GJ}$

or $T_A = 3T_B$ or $T_A = \frac{3T}{4}$,

$\tau_{max} = \frac{16T_A}{\pi D3} = \frac{16 \times \frac{3}{4} \times T}{\pi D_3^4} = \frac{12T}{\pi D3}$

IES-44. **Two shafts are shown in the above figure. These two shafts will be torsionally equivalent to each other if their**

(a) Polar moment of inertias are the same
(b) Total angle of twists are the same
(c) Lengths are the same
(d) Strain energies are the same



[IES-1998]

IES-44. Ans. (b)

Previous 20-Years IAS Questions

Torsion Equation

IAS-1. **Assertion (A): In theory of torsion, shearing strains increase radically away from the longitudinal axis of the bar.** **[IAS-2001]**
Reason (R): Plane transverse sections before loading remain plane after the torque is applied.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-1. Ans. (b)

IAS-2. The shear stress at a point in a shaft subjected to a torque is: [IAS-1995]
(a) Directly proportional to the polar moment of inertia and to the distance of the point form the axis
(b) Directly proportional to the applied torque and inversely proportional to the polar moment of inertia.
(c) Directly proportional to the applied torque and polar moment of inertia
(d) inversely proportional to the applied torque and the polar moment of inertia

IAS-2. Ans. (b) $\frac{T}{J}=\frac{\tau}{R}$

IAS-3. If two shafts of the same length, one of which is hollow, transmit equal torque and have equal maximum stress, then they should have equal. [IAS-1994]
(a) Polar moment of inertia (b) Polar modulus of section
(c) Polar moment of inertia (d) Angle of twist

IAS-3. Ans. (b) $\frac{T}{J}=\frac{\tau}{R}$ Here T & τ are same, so $\frac{J}{R}$ should be same i.e. polar modulus of section will be same.

Hollow Circular Shafts

IAS-4. A hollow circular shaft having outside diameter 'D' and inside diameter 'd' subjected to a constant twisting moment 'T' along its length. If the maximum shear stress produced in the shaft is σ_s then the twisting moment 'T' is given by: [IAS-1999]

(a) $\frac{\pi}{8}\sigma_s\frac{D^4-d^4}{D^4}$ (b) $\frac{\pi}{16}\sigma_s\frac{D^4-d^4}{D^4}$ (c) $\frac{\pi}{32}\sigma_s\frac{D^4-d^4}{D^4}$ (d) $\frac{\pi}{64}\sigma_s\frac{D^4-d^4}{D^4}$

IAS-4. Ans. (b) $\frac{T}{J}=\frac{G\theta}{L}=\frac{\tau}{R}$ gives $T=\frac{\tau J}{R}=\frac{\sigma_s\times\frac{\pi}{32}\left(D^4-d^4\right)}{\frac{D}{2}}=\frac{\pi}{16}\sigma_s\frac{\left(D^4-d^4\right)}{D}$

Torsional Rigidity

IAS-5. Match List-I with List-II and select the correct answer using the codes given below the lists: [IAS-1996]

List-I (Mechanical Properties)	**List-II (Characteristics)**
A. Torsional rigidity	**1.** Product of young's modulus and second moment of area about the plane of bending
B. Modulus of resilience	**2.** Strain energy per unit volume
C. Bauschinger effect	**3.** Torque unit angle of twist
D. Flexural rigidity	**4.** Loss of mechanical energy due to local yielding

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	1	3	4	2	(b)	3	2	4	1
(c)	2	4	1	3	(d)	3	1	4	2

IAS-5. Ans. (b)

IAS-6. Assertion (A): Angle of twist per unit length of a uniform diameter shaft depends upon its torsional rigidity. [IAS-2004]
Reason (R): The shafts are subjected to torque only.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A

(c) A is true but R is false
(d) A is false but R is true

IAS-6. Ans. (c)

Combined Bending and Torsion

IAS-7. A shaft is subjected to a bending moment M = 400 N.m alld torque T = 300 N.m The equivalent bending moment is: [IAS-2002]

(a) 900 N.m (b) 700 N.m (c) 500 N.m (d) 450 N.m

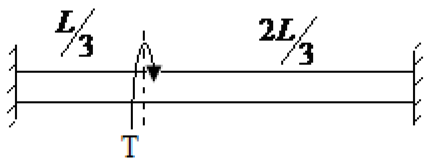
IAS-7. Ans. (d) $Me = \dfrac{M + \sqrt{M^2 + T^2}}{2} = \dfrac{400 + \sqrt{400^2 + 300^2}}{2} = 450 Nm$

Comparison of Solid and Hollow Shafts

IAS-8. A hollow shaft of length L is fixed at its both ends. It is subjected to torque T at a distance of $\dfrac{L}{3}$ from one end. What is the reaction torque at the other end of the shaft? [IAS-2007]

(a) $\dfrac{2T}{3}$ (b) $\dfrac{T}{2}$ (c) $\dfrac{T}{3}$ (d) $\dfrac{T}{4}$

IAS-8. Ans. (c)



IAS-9. A solid shaft of diameter d is replaced by a hollow shaft of the same material and length. The outside diameter of hollow shaft $\dfrac{2d}{\sqrt{3}}$ while the inside diameter is $\dfrac{d}{\sqrt{3}}$. What is the ratio of the torsional stiffness of the hollow shaft to that of the solid shaft? [IAS-2007]

(a) $\dfrac{2}{3}$ (b) $\dfrac{3}{5}$ (c) $\dfrac{5}{3}$ (d) 2

IAS-9. Ans. (c) Torsional stiffness $= \left(\dfrac{T}{\theta}\right) = \dfrac{GJ}{L}$ or $\dfrac{K_H}{K_S} = \dfrac{\dfrac{\pi}{32}\left\{\left(\dfrac{2d}{\sqrt{3}}\right)^4 - \left(\dfrac{d}{\sqrt{3}}\right)^4\right\}}{\dfrac{\pi}{32}.d^4} = \dfrac{5}{3}$

IAS-10. Two steel shafts, one solid of diameter D and the other hollow of outside diameter D and inside diameter D/2, are twisted to the same angle of twist per unit length. The ratio of maximum shear stress in solid shaft to that in the hollow shaft is: [IAS-1998]

(a) $\dfrac{4}{9}\tau$ (b) $\dfrac{8}{7}\tau$ (c) $\dfrac{16}{15}\tau$ (d) τ

IAS-10. Ans. (d) $\dfrac{T}{J} = \dfrac{\tau}{R} = \dfrac{G\theta}{L}$ or $\tau = \dfrac{G\theta R}{L}$ as outside diameter of both the shaft is D so τ is same for both the cases.

Shafts in Series

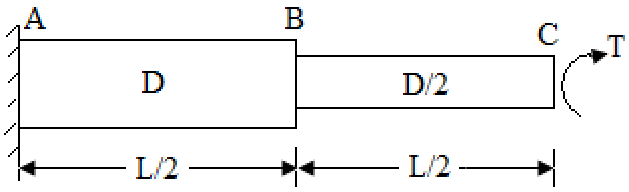
IAS-11. Two shafts having the same length and material are joined in series. If the ratio of the diameter of the first shaft to that of the second shaft is 2, then the ratio of the angle of twist of the first shaft to that of the second shaft is: **[IAS-1995; 2003]**

(a) 16 (b) 8 (c) 4 (d) 2

IAS-11. Ans. (a) Angle of twist is proportional to $\frac{1}{J} \propto \frac{1}{d^4}$

IAS-12. A circular shaft fixed at A has diameter D for half of its length and diameter D/2 over the other half. What is the rotation of C relative of B if the rotation of B relative to A is 0.1 radian? **[IAS-1994]**

(a) 0.4 radian (b) 0.8 radian (c) 1.6 radian (d) 3.2 radian



(T, L and C remaining same in both cases)

IAS-12. Ans. (c) $\frac{T}{J} = \frac{G\theta}{L}$ or $\theta \propto \frac{1}{J}$ or $\theta \propto \frac{1}{d^4}$ $\because J = \frac{\pi d^4}{32}$

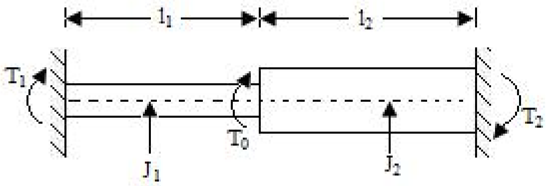
Here $\frac{\theta}{0.1} = \frac{d^4}{(d/2)^4}$ or $\theta = 1.6$ radian.

Shafts in Parallel

IAS-13. A stepped solid circular shaft shown in the given figure is built-in at its ends and is subjected to a torque T_o at the shoulder section. The ratio of reactive torque T_1 and T_2 at the ends is (J_1 and J_2 are polar moments of inertia):

(a) $\frac{J_2 \times l_2}{J_1 \times l_1}$ (b) $\frac{J_2 \times l_1}{J_1 \times l_2}$

(c) $\frac{J_1 \times l_2}{J_2 \times l_1}$ (d) $\frac{J_1 \times l_1}{J_2 \times l_2}$



[IAS-2001]

IAS-13. Ans. (c) $\theta_1 = \theta_2$ or $\frac{T_1 l_1}{GJ_1} = \frac{T_2 l_2}{GJ_2}$ or $\frac{T_1}{T_2} = \left(\frac{J_1}{J_2} \times \frac{l_2}{l_1}\right)$

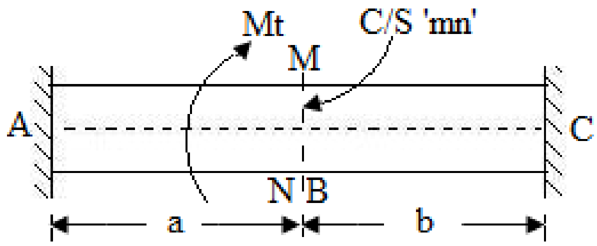
IAS-14. Steel shaft and brass shaft of same length and diameter are connected by a flange coupling. The assembly is rigidity held at its ends and is twisted by a torque through the coupling. Modulus of rigidity of steel is twice that of brass. If torque of the steel shaft is 500 Nm, then the value of the torque in brass shaft will be: **[IAS-2001]**

(a) 250 Nm (b) 354 Nm (c) 500 Nm (d) 708 Nm

IAS-14. Ans. (a)

$$\theta_1 = \theta_2 \; or \; \frac{T_s l_s}{G_s J_s} = \frac{T_b l_b}{G_b J_b} \quad or \; \frac{T_s}{G_s} = \frac{T_b}{G_b} \quad or \; \frac{T_b}{T_s} = \frac{G_b}{G_s} = \frac{1}{2} \quad or \; T_b = \frac{T_s}{2} = 250\,\text{Nm}$$

IAS-15. A steel shaft with bult-in ends is subjected to the action of a torque Mt applied at an intermediate cross-section 'mn' as shown in the given figure. **[IAS-1997]**



Assertion (A): The magnitude of the twisting moment to which the portion BC is subjected is $\dfrac{M_t a}{a+b}$

Reason(R): For geometric compatibility, angle of twist at 'mn' is the same for the portions AB and BC.

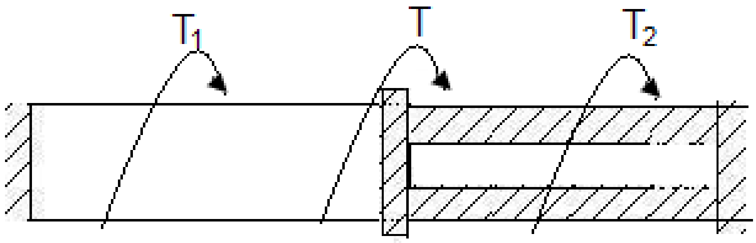
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-15. Ans. (a)

IAS-16. A steel shaft of outside diameter 100 mm is solid over one half of its length and hollow over the other half. Inside diameter of hollow portion is 50 mm. The shaft if held rigidly at two ends and a pulley is mounted at its midsection i.e., at the junction of solid and hollow portions. The shaft is twisted by applying torque on the pulley. If the torque carried by the solid portion of the shaft is 16000kg-m, then the torque carried by the hollow portion of the shaft will be: [IAS-1997]

(a) 16000 kg-m (b) 15000 kg-m (c) 14000 kg-m (d) 12000 kg-m

IAS-16. Ans.(b) $\theta_s = \theta_H$ or $\dfrac{T_s L}{GJ_s} = \dfrac{T_H L}{GJ_H}$ or $T_H = T_S \times \dfrac{J_H}{J_s} = 16000 \times \dfrac{\frac{\pi}{32}\left(100^4 - 50^4\right)}{\frac{\pi}{32}\left(100^4\right)} = 15000\,\text{kgm}$



Previous Conventional Questions with Answers

Conventional Question IES 2010

Q. **A hollow steel rod 200 mm long is to be used as torsional spring. The ratio of inside to outside diameter is 1 : 2. The required stiffness of this spring is 100 N.m /degree.**
Determine the outside diameter of the rod.
Value of G is $8\times10^4\ \text{N/mm}^2$. **[10 Marks]**

Ans. Length of a hollow steel rod = 200mm
Ratio of inside to outside diameter = 1 : 2
Stiffness of torsional spring = 100 Nm /degree. = 5729.578 N m/rad
Rigidity of modulus (G) = $8\times10^4\ \text{N/mm}^2$
Find outside diameter of rod : -
We know that

$$\frac{T}{J}=\frac{G.\theta}{L}$$ Where T = Torque

$$\frac{\mathbf{T}}{\boldsymbol{\theta}}=\textbf{Stiffness}\left(\frac{\mathbf{N-M}}{\mathbf{rad}}\right)$$

J = polar moment

Stiffness = $\frac{T}{\theta}=\frac{G.J}{L}$

$\boldsymbol{\theta}$ = twist angle in rad

L = length of rod.

$$\mathbf{d_2 = 2d_1}$$

$$J=\frac{\pi}{32}\times\left(d_2^4 - d_1^4\right)$$

$$J=\frac{\pi}{32}\times\left(16d_1^4 - d_1^4\right) \qquad \because \frac{d_1}{d_2}=\frac{1}{2}$$

$$J=\frac{\pi}{32}\times d_1^4\times15$$

$$5729.578\,\text{Nm/rad}=\frac{8\times10^4\times10^6\ \text{N/m}^2}{0.2}\times\frac{\pi}{32}\times d_1^4\times15$$

$$\frac{5729.578\times.2\times32}{8\times10^{10}\times\pi\times15}=d_1^4$$

$$d_1=9.93\times10^{-3}\ \text{m.}$$

$$d_1=9.93\,\text{mm.}$$

$$d_2=2\times9.93=19.86\ \text{mm} \qquad \text{Ans.}$$

Conventional Question GATE - 1998

***Question*:** **A component used in the Mars pathfinder can be idealized as a circular bar clamped at its ends. The bar should withstand a torque of 1000 Nm. The component is assembled on earth when the temperature is 30°C. Temperature on Mars at the site of landing is -70°C. The material of the bar has an allowable shear stress of 300 MPa and its young's modulus is 200 GPa. Design the diameter of the bar taking a factor of safety of 1.5 and assuming a coefficient of thermal expansion for the material of the bar as 12×10^{-6}/°C.**

***Answer*:** Given:

$T_{max} = 1000Nm; \quad t_E = 30^0C; \; t_m = -70^0C; \quad \tau_{allowable} = 300MPa$

$E = 200GPa; \quad F.O.S. = 1.5; \quad \alpha = 12\times10^{-6} /^0 C$

Diameter of the bar, D :

Change in length, $\delta L = L \propto \Delta t$, where L = original length, m.

Change in length at Mars $= L \times 12 \times 10^{-6} \times [30 - (-70)] = 12\times10^{-4}L$ meters

$$\text{Linear strain} = \frac{\text{Change in length}}{\text{original length}} = \frac{12\times10^{-4}L}{L} = 12\times10^{-4}$$

$$\sigma_a = \text{axial stress} = E \times \text{linear strain} = 200\times10^9 \times 12\times10^{-4} = 2.4\times10^8 N/m^2$$

From maximum shear stress equation, we have

$$\tau_{max} = \sqrt{\left[\left(\frac{16T}{\pi D^3}\right)^2 + \left(\frac{\sigma_a}{2}\right)^2\right]}$$

where, $\tau_{max} = \dfrac{\tau_{allowable}}{F.O.S} = \dfrac{300}{1.5} = 200MPa$

Substituting the values, we get

$$4\times10^{16} = \left(\frac{16\times1000}{\pi D^3}\right)^2 + \left(1.2\times10^8\right)^2$$

or $\dfrac{16\times1000}{\pi D^3} = 1.6\times10^8$

or $D = \left(\dfrac{16\times1000}{\pi\times1.6\times10^8}\right)^{1/3} = 0.03169 \text{ m} = 31.69 \text{ mm}$

Conventional Question IES-2009

Q. **In a torsion test, the specimen is a hollow shaft with 50 mm external and 30 mm internal diameter. An applied torque of 1.6 kN-m is found to produce an angular twist of 0.4º measured on a length of 0.2 m of the shaft. The Young's modulus of elasticity obtained from a tensile test has been found to be 200 GPa. Find the values of**

(i) Modulus of rigidity.

(ii) Poisson's ratio. **[10-Marks]**

Ans. We have

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \quad \text{......... (i)}$$

Where J = polar moment of inertia

$$J = \frac{\pi}{32}\left(D^4 - d^4\right)$$

$$= \frac{\pi}{32}\left(50^4 - 30^4\right)\times10^{-12}$$

$$= 5.338\times10^{-7}$$

$T = 1.6 \text{ kN} - \text{m} = 1.6\times10^3$ **N-m**

$\theta = 0.4^o$

$l = 0.2$ **m**

$E = 200 \times 10^9$ **N/m²**

From equation (i) $\dfrac{T}{J} = \dfrac{G\theta}{L}$

$$\frac{1.6\times10^{3}}{5.338\times10^{-7}}=\frac{G\times\left[0.4\times\frac{\pi}{180}\right]}{0.2}$$

$$\Rightarrow\ G=\frac{1.6\times0.2\times10^{3}\times180}{0.4\times\pi\times5.338\times10^{-7}}$$

$$=85.92\ \text{GPa}$$

We also have

$$E=2G(1+v)$$

$$\therefore\ 200=2\times85.92\,(1+v)$$

$$\Rightarrow\ 1+v=1.164$$

$$\Rightarrow\ v=0.164$$

Conventional Question IAS - 1996

***Question*:** **A solid circular uniformly tapered shaft of length I, with a small angle of taper is subjected to a torque T. The diameter at the two ends of the shaft are D and 1.2 D. Determine the error introduced of its angular twist for a given length is determined on the uniform mean diameter of the shaft.**

***Answer*:** For shaft of tapering's section, we have

$$\theta=\frac{2TL}{3G\pi}\left[\frac{R_1^2+R_1R_2+R_2^2}{R_1^3R_2^3}\right]=\frac{32TL}{3G\pi}\left[\frac{D_1^2+D_1D_2+D_2^2}{D_1^3D_2^3}\right]$$

$$=\frac{32TL}{3G\pi D^4}\left[\frac{(1.2)^2+1.2\times1+(1)^2}{(1.2)^3\times(1)^3}\right]\qquad[\because D_1=D\ \text{ and }\ D_2=1.2D]$$

$$=\frac{32TL}{3G\pi D^4}\times2.1065$$

Now, $D_{avg}=\dfrac{1.2D+D}{2}=1.1D$

$$\therefore\qquad \theta'=\frac{32TL}{3G\pi}\times\left[\frac{3(1.1D)^2}{(1.1D)^6}\right]=\frac{32TL}{3G\pi}\times\frac{3}{(1.2)^4.D^4}=\frac{32TL}{3G\pi D^4}\times2.049$$

$$\text{Error}=\frac{\theta-\theta'}{\theta}=\frac{2.1065-2.049}{2.1065}=0.0273\ \text{or}\ 2.73\%$$

Conventional Question ESE-2008

***Question*:** **A hollow shaft and a solid shaft construction of the same material have the same length and the same outside radius. The inside radius of the hollow shaft is 0.6 times of the outside radius. Both the shafts are subjected to the same torque.**

(i) What is the ratio of maximum shear stress in the hollow shaft to that of solid shaft?

(ii) What is the ratio of angle of twist in the hollow shaft to that of solid shaft?

***Solution*:** Using $\dfrac{T}{J}=\dfrac{\tau}{R}=\dfrac{G\theta}{L}$

Given, $\dfrac{\text{Inside radius (r)}}{\text{Out side (R)}}=0.6$ and $T_h=T_s=T$

(i) $\tau=\dfrac{T.R}{J}$ gives ; For hollow shaft $(\tau_h)=\dfrac{T.R}{\frac{\pi}{2}\left(R^4-r^4\right)}$

and for solid shaft (τ_s)= $\dfrac{T.R}{\frac{\pi}{2}.R^4}$

Therefore $\dfrac{\tau_n}{\tau_s} = \dfrac{R^4}{R^4 - r^4} = \dfrac{1}{1-\left(\frac{r}{R}\right)^4} = \dfrac{1}{1-0.6^4} = 1.15$

(ii) $\theta = \dfrac{TL}{GJ}$ gives $\theta_h = \dfrac{T.L}{G.\frac{\pi}{2}\left(R^4 - r^4\right)}$ and $\theta_s = \dfrac{T.L}{G.\left(\frac{\pi}{2}.R^4\right)}$

Therefore $\dfrac{\theta_h}{\theta_s} = \dfrac{R^4}{R^4 - r^4} = \dfrac{1}{1-\left(\frac{r}{R}\right)^4} = \dfrac{1}{1-0.6^4} = 1.15$

Conventional Question ESE-2006:

Question: **Two hollow shafts of same diameter are used to transmit same power. One shaft is rotating at 1000 rpm while the other at 1200 rpm. What will be the nature and magnitude of the stress on the surfaces of these shafts? Will it be the same in two cases of different? Justify your answer.**

Answer: We know power transmitted (P) = Torque (T) ×rotation speed (ω)

And shear stress (τ) = $\dfrac{T.R}{J} = \dfrac{PR}{\omega J} = \dfrac{P.D/2}{\left(\frac{2\pi N}{60}\right)\frac{\pi}{32}\left(D^4 - d^4\right)}$

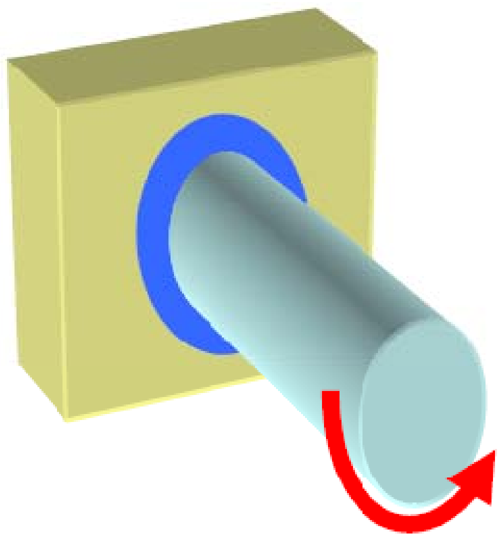
Therefore $\tau \; \alpha \; \dfrac{1}{N}$ as P, D and d are constant.

So the shaft rotating at 1000 rpm will experience greater stress then 1200 rpm shaft.

Conventional Question ESE-2002

Question: **A 5 cm diameter solid shaft is welded to a flat plate by 1 cm filled weld. What will be the maximum torque that the welded joint can sustain if the permissible shear stress in the weld material is not to exceed 8 kN/cm²? Deduce the expression for the shear stress at the throat from the basic theory.**

Answer: Consider a circular shaft connected to a plate by means of a fillet joint as shown in figure. If the shaft is subjected to a torque, shear stress develops in the weld. Assuming that the weld thickness is very small compared to the diameter of the shaft, the maximum shear stress occurs in the throat area. Thus, for a given torque the maximum shear stress in the weld is



$$\tau_{max} = \frac{T\left(\frac{d}{2}+t\right)}{J}$$

Where T = Torque applied.
d = outer diameter of the shaft
t = throat thickness
J = polar moment of area of the throat section

$$= \frac{\pi}{32}\left[(d+2t)^4 - d^4\right] = \frac{\pi}{4}d^3 \times t$$

[As t <<d] then $\tau_{\max} = \dfrac{T\dfrac{d}{2}}{\dfrac{\pi}{4}d^3t} = \dfrac{2T}{\pi t d^2}$

Given

d = 5 cm = 0.05 m & t = 1cm = 0.1 m

$$\tau_{\max} = 8kN/cm^2 = \frac{8000N}{\left(10^{-4}\right)m^2} = 80MPa = 80\times10^6 N/m^2$$

$$\therefore T = \frac{\pi d^2 t\tau_{\max}}{2} = \frac{\pi\times0.05^2\times0.01\times80\times10^6}{2} = 3.142\,kNm$$

Conventional Question ESE-2000

Question: **The ratio of inside to outside diameter of a hollow shaft is 0.6. If there is a solid shaft with same torsional strength, what is the ratio of the outside diameter of hollow shaft to the diameter of the equivalent solid shaft.**

Answer: Let D = external diameter of hollow shaft

So d = 0.6D internal diameter of hollow shaft

And D_s=diameter of solid shaft

From torsion equation

$$\frac{T}{J} = \frac{\tau}{R}$$

$$or, T = \frac{\tau J}{R} = \tau\times\frac{\frac{\pi}{32}\{D^4 - (0.6D)^4\}}{(D/2)}\text{ for hollow shaft}$$

$$and\ \ T = \frac{\tau J}{R} = J\times\frac{\frac{\pi}{32}D_s^4}{D_s/2}\text{ for solid shaft}$$

$$\tau\frac{\pi D^3}{16}\{1-(0.6)^4\} = \tau\frac{\pi D_s^3}{16}$$

$$or, \frac{D}{D_s} = \sqrt[3]{\frac{1}{1-(0.6)^4}} = 1.072$$

Conventional Question ESE-2001

Question: **A cantilever tube of length 120 mm is subjected to an axial tension P = 9.0 kN, A torsional moment T = 72.0 Nm and a pending Load F = 1.75 kN at the free end. The material is aluminum alloy with an yield strength 276 MPa. Find the thickness of the tube limiting the outside diameter to 50 mm so as to ensure a factor of safety of 4.**

Answer: Polar moment of inertia (J) $=2\pi R^3 t = \dfrac{\pi D^3 t}{4}$

$$\frac{T}{J}=\frac{\tau}{R} \text{ or, } \tau=\frac{T.R}{J}=\frac{TD}{2J}=\frac{TD}{2\times\frac{\pi D^3 t}{4}}=\frac{2T}{\pi D^2 t}=\frac{2\times 72}{\pi\times(0.050)^2\times t}=\frac{18335}{t}$$

$$\text{Direct stress } (\sigma_1)=\frac{P}{A}=\frac{9000}{\pi dt}=\frac{9000}{\pi(0.050)t}=\frac{57296}{t}$$

$$\text{Maximum bending stress } (\sigma_2)=\frac{My}{I}=\frac{M\,d/2}{I}=\frac{Md}{J} \qquad [J=2I]$$

$$=\frac{1750\times 0.120\times 0.050\times 4}{\pi\times(0.050)^3 t}=\frac{106952}{t}$$

$$\therefore \text{Total longitudinal stress } (\sigma_b)=\sigma_1+\sigma_2=\frac{164248}{t}$$

Maximum principal stress

$$(\sigma_1)=\frac{\sigma_b}{2}+\sqrt{\left(\frac{\sigma_b}{2}\right)^2+\tau^2}=\frac{164248}{2t}+\sqrt{\left(\frac{164248}{2t}\right)^2+\left(\frac{18335}{t}\right)^2}=\left(\frac{276\times 10^6}{4}\right)$$

$$or, t=2.4\times 10^{-3} m=2.4\,mm$$

Conventional Question ESE-2000 & ESE 2001

Question: **A hollow shaft of diameter ratio 3/8 required to transmit 600 kW at 110 rpm, the maximum torque being 20% greater than the mean. The shear stress is not to exceed 63 MPa and the twist in a length of 3 m not to exceed 1.4 degrees. Determine the diameter of the shaft. Assume modulus of rigidity for the shaft material as 84 GN/m².**

Answer: Let d = internal diameter of the hollow shaft

And D = external diameter of the hollow shaft

(given) d = 3/8 D = 0.375D

Power (P)= 600 kW, speed (N) =110 rpm, Shear stress(τ)= 63 MPa. Angle of twist (θ)=1.4°, Length (ℓ) =3m , modulus of rigidity (G) = 84GPa

We know that, (P) = T. ω = T. $\frac{2\pi N}{60}$ [T is average torque]

$$\text{or } T=\frac{60\times P}{2\pi N}=\frac{60\times(600\times 10^3)}{2\times\pi\times 110}=52087\text{Nm}$$

$$\therefore T_{max}=1.2\times T=1.2\times 52087 = 62504 \text{ Nm}$$

First we consider that shear stress is not to exceed 63 MPa

From torsion equation $\frac{T}{J}=\frac{\tau}{R}$

$$or\ J=\frac{T.R}{\tau}=\frac{T.D}{2\tau}$$

$$or\ \frac{\pi}{32}\left[D^4-(0.375D)^4\right]=\frac{62504\times D}{2\times(63\times 10^6)}$$

$$or\ D=0.1727m=172.7mm----(i)$$

Second we consider angle of twist is not exceed $1.4^0=\frac{17\times 1.4}{180}$ radian

From torsion equation $\frac{T}{J}=\frac{G\theta}{\ell}$

$$or\ \frac{T}{J}=\frac{G\theta}{\ell}$$

$$or\ \frac{\pi}{32}\left[D^4-(0.375D)^4\right]=\frac{62504\times 3}{(84\times 10^9)\left(\frac{\pi\times 1.5}{180}\right)}$$

$$or\ D=0.1755m=175.5mm----(ii)$$

So both the condition will satisfy if greater of the two value is adopted

so D=175.5 mm

Conventional Question ESE-1997

Question: **Determine the torsional stiffness of a hollow shaft of length L and having outside diameter equal to 1.5 times inside diameter d. The shear modulus of the material is G.**

Answer: Outside diameter (D) =1.5 d

Polar modulus of the shaft (J) = $\frac{\pi}{32}\left(D^4-d^4\right)=\frac{\pi}{32}d^4(1.5^4-1)$

We know that $\frac{T}{J}=\frac{\tau}{R}=\frac{G\theta}{L}$

$$or\ T=\frac{G\theta J}{L}=\frac{G.\theta\frac{\pi}{32}d^4(1.5^4-1)}{L}=\frac{0.4G\theta d^4}{L}$$

Conventional Question AMIE-1996

Question: **The maximum normal stress and the maximum shear stress analysed for a shaft of 150 mm diameter under combined bending and torsion, were found to be 120 MN/m² and 80 MN/m² respectively. Find the bending moment and torque to which the shaft is subjected.**

If the maximum shear stress be limited to 100 MN/m², find by how much the torque can be increased if the bending moment is kept constant.

Answer: Given: $\sigma_{max}=120\,MN/m^2;\tau_{max}=80\,MN/m^2;d=150\,mm=0.15\,m$

Part – 1: M; T

We know that for combined bending and torsion, we have the following expressions:

$$\sigma_{max}=\frac{16}{\pi d^3}\left[M+\sqrt{M^2+T^2}\right]\qquad ---(i)$$

and $$\tau_{max}=\frac{16}{\pi d^3}\left[\sqrt{M^2+T^2}\right]\qquad ----(ii)$$

Substituting the given values in the above equations, we have

$$120=\frac{16}{\pi\times(0.15)^3}\left[M+\sqrt{M^2+T^2}\right]------(iii)$$

$$80=\frac{16}{\pi\times(0.15)^3}\left[\sqrt{M^2+T^2}\right]----------(iv)$$

or $$\sqrt{M^2+T^2}=\frac{80\times\pi\times(0.15)^3}{16}=0.053------(v)$$

Substituting this values in equation (iii), we get

$$120 = \frac{16}{\pi \times (0.150^3)}[M + 0.053]$$

$\therefore$ $M = 0.0265\,MNm$

Substituting for M in equation (v), we have

$$\sqrt{(0.0265)^2 + T^2} = 0.053$$

or $T = 0.0459\,MNm$

Part II: $[\because \tau_{max} = 100MN/m^2]$

Increase in torque:

Bending moment (M) to be kept constant $= 0.0265MNm$

or $$(0.0265)^2 + T^2 = \left[\frac{100 \times \pi \times (0.15)^3}{16}\right]^2 = 0.004391$$

$\therefore$ $T = 0.0607\ MNm$

$\therefore$ The increased torque $= 0.0607 - 0.0459 = 0.0148\,MNm$

Conventional Question ESE-1996

Question: **A solid shaft is to transmit 300 kW at 120 rpm. If the shear stress is not to exceed 100 MPa, Find the diameter of the shaft, What percent saving in weight would be obtained if this shaft were replaced by a hollow one whose internal diameter equals 0.6 of the external diameter, the length, material and maximum allowable shear stress being the same?**

Answer: Given P= 300 kW, N = 120 rpm, τ =100 MPa, $d_H = 0.6D_H$

Diameter of solid shaft, D_s:

We know that $P = \frac{2\pi NT}{60 \times 1000}$ or $300 = \frac{2\pi \times 120 \times T}{60 \times 1000}$ or T=23873 Nm

We know that $\frac{T}{J} = \frac{\tau}{R}$

or, $T = \frac{\tau . J}{R}$ or, $23873 = \frac{100 \times 10^6 \times \frac{\pi}{32} D_s^4}{\frac{D_s}{2}}$

or, D_s= 0.1067 m =106.7mm

Percentage saving in weight:

$T_H = T_s$

$$\left(\frac{\tau \times J}{R}\right)_H = \left(\frac{\tau \times J}{R}\right)_s$$

$$or, \frac{\{D_H^4 - d_H^4\}}{D_H} = D_s^3 \qquad or, \frac{D_H^4 - (0.6D_H)^4}{D_H} = D_s^3$$

$$or, D_H = \frac{D_s}{\sqrt[3]{(1-0.6^4)}} = \frac{106.7}{\sqrt[3]{1-0.64}} = 111.8\,mm$$

$$\text{Again } \frac{W_H}{W_S} = \frac{A_H L_H \rho_H g}{A_s L_s \rho_s g} = \frac{A_H}{A_s}$$

$$\frac{A_H}{A_s} = \frac{\frac{\pi}{4}\left(D_H^2 - d_H^2\right)}{\frac{\pi}{4}D_s^2} = \frac{D_H^2(1-0.6^2)}{D_s^2} = \left(\frac{111.8}{106.7}\right)^2 (1-0.6)^2 = 0.702$$

$$\therefore \text{Percentage savings in weight} = \left(1 - \frac{W_H}{W_s}\right) \times 100$$

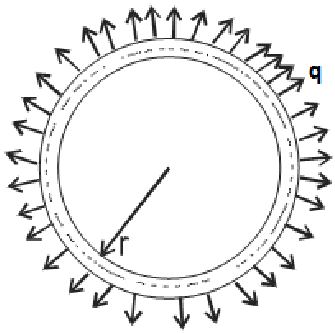
$$= (1-0.702) \times 100 = 29.8\%$$

10. Thin Cylinder

Theory at a Glance (for IES, GATE, PSU)

1. Thin Rings

Uniformly distributed loading (radial) may be due to either



- Internal pressure or external pressure
- Centrifugal force as in the case of a rotating ring

Case-I: Internal pressure or external pressure

- s = qr Where q = Intensity of loading in kg/cm of O[ce]
  r = Mean centreline of radius
  s = circumferential tension or hoop's tension
  (Radial loading ducted outward)
- Unit stress, $\sigma = \dfrac{s}{A} = \dfrac{qr}{A}$
- Circumferential strain, $\in_c = \dfrac{\sigma}{\mathrm{E}} = \dfrac{qr}{AE}$
- Diametral strain, $(\in_d)$ = Circumferential strain, $(\in_c)$

Case-II: Centrifugal force

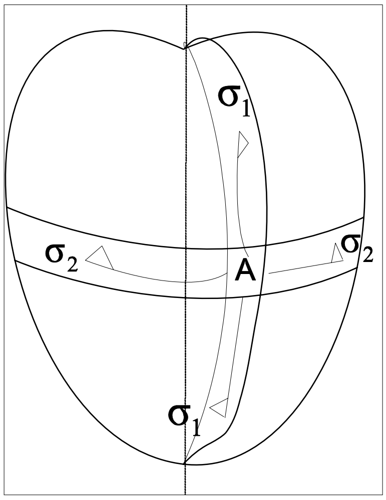
- Hoop's Tension, $s = \dfrac{w\omega^2 r^2}{g}$ Where w = wt. per unit length of circumferential element
  ω = Angular velocity
- Radial loading, $q = \dfrac{s}{r} = \dfrac{w\omega^2 r}{g}$
- Hoop's stress, $\sigma = \dfrac{s}{A} = \dfrac{w}{Ag}.\omega^2 r^2$

2. Thin Walled Pressure Vessels

For thin cylinders whose thickness may be considered small compared to their diameter.

$$\frac{\text{Inner dia of the cylinder } (\mathrm{d_i})}{\text{wall thickness } (\mathrm{t})} > 15 \text{ or } 20$$

3. General Formula



$$\frac{\sigma_1}{r_1}+\frac{\sigma_2}{r_2}=\frac{p}{t}$$

Where σ_1 =Meridional stress at A

σ_2 =Circumferential / Hoop's stress

P = Intensity of internal gas pressure/ fluid pressure

t = Thickness of pressure vessel.

4. Some cases:

- ***Cylindrical vessel***

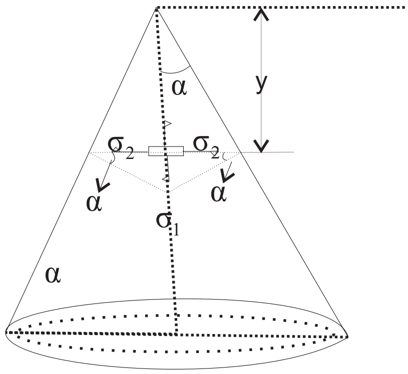
$$\sigma_1=\frac{pr}{2t}=\frac{pD}{4t} \qquad \sigma_2=\frac{pr}{t}=\frac{pD}{2t} \qquad \left[r_1\to\infty, r_2=r\right]$$

$$\tau_{\max}=\frac{\sigma_1-\sigma_2}{2}=\frac{pr}{4t}=\frac{pD}{8t}$$

- ***Spherical vessel***

$$\sigma_1=\sigma_2=\frac{pr}{2t}=\frac{pD}{4t} \qquad [r_1=r_2=r]$$

- ***Conical vessel***



$$\sigma_1=\frac{py\tan\alpha}{2t\cos\alpha}[r_1\to\infty] \quad \text{and} \quad \sigma_2=\frac{py\tan\alpha}{t\cos\alpha}$$

Notes:

- Volume 'V' of the spherical shell, $V=\frac{\pi}{6}D_i^3$

$$\Rightarrow D_i=\left(\frac{6V}{\pi}\right)^{1/3}$$

- Design of thin cylindrical shells is based on hoop's stress

5. Volumetric Strain (Dilation)

- Rectangular block, $\frac{\Delta V}{V_0}=\in_x+\in_y+\in_z$
- ***Cylindrical pressure vessel***

$\in_1$=Longitudinal strain $=\frac{\sigma_1}{E}-\mu\frac{\sigma_2}{E}=\frac{pr}{2Et}[1-2\mu]$

$\in_2$ =Circumferential strain $=\frac{\sigma_2}{E}-\mu\frac{\sigma_1}{E}=\frac{pr}{2Et}[1-2\mu]$

Volumetric Strain, $\frac{\Delta V}{V_o}=\in_1+2\in_2=\frac{pr}{2Et}[5-4\mu]=\frac{pD}{4Et}[5-4\mu]$

i.e. *Volumetric strain*, $(\in_v)$ = *longitudinal strain* $(\in_1)+2\times$ *circumferential strain* $(\in_2)$

- ***Spherical vessels***

$$\in=\in_1=\in_2=\frac{pr}{2Et}[1-\mu]$$

$$\frac{\Delta V}{V_0} = 3 \in = \frac{3pr}{2Et}[1-\mu]$$

6. Thin cylindrical shell with hemispherical end

Condition for no distortion at the junction of cylindrical and hemispherical portion

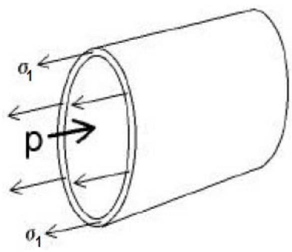
$$\frac{t_2}{t_1} = \frac{1-\mu}{2-\mu}$$

Where, t_1= wall thickness of cylindrical portion

t_2 = wall thickness of hemispherical portion

7. Alternative method

Consider the equilibrium of forces in the z-direction acting on the part cylinder shown in figure.



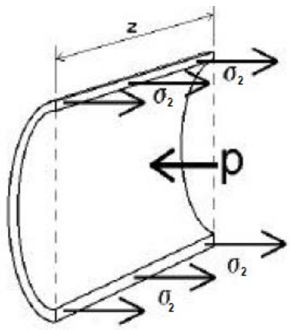
Force due to internal pressure p acting on area π D²/4 = p. π D²/4

Force due to longitudinal stress sL acting on area π Dt = $\sigma_1 \pi$ Dt

Equating: p. π D²/4 = $\sigma_1 \pi$ Dt

or $$\sigma_1 = \frac{pd}{4t} = \frac{pr}{2t}$$

Now consider the equilibrium of forces in the x-direction acting on the sectioned cylinder shown in figure. It is assumed that the circumferential stress σ_2 is constant through the thickness of the cylinder.



Force due to internal pressure p acting on area Dz = pDz

Force due to circumferential stress σ_2 acting on area 2tz = σ_2 2tz

Equating: pDz = σ_2 2tz

or $$\sigma_2 = \frac{pD}{2t} = \frac{pr}{t}$$

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Longitudinal stress

GATE-1. The maximum principal strain in a thin cylindrical tank, having a radius of 25 cm and wall thickness of 5 mm when subjected to an internal pressure of 1MPa, is (taking Young's modulus as 200 GPa and Poisson's ratio as 0.2) [GATE-1998]

(a) 2.25×10^{-4} (b) 2.25 (c) 2.25×10^{-6} (d) 22.5

GATE-1. Ans. (a) Circumferential or Hoop stress $(\sigma_c) = \dfrac{pr}{t} = \dfrac{1\times 250}{5} = 50\text{MPa}$

Longitudinal stress $(\sigma_l) = \dfrac{pr}{2t} = 25\text{MPa}$

$$e_c = \frac{\sigma_c}{E} - \mu\frac{\sigma_l}{E} = \frac{50\times 10^6}{200\times 10^9} - 0.2\times\frac{25\times 10^6}{200\times 10^9} = 2.25\times 10^{-4}$$

Maximum shear stress

GATE-2. A thin walled cylindrical vessel of well thickness, t and diameter d is fitted with gas to a gauge pressure of p. The maximum shear stress on the vessel wall will then be: [GATE-1999]

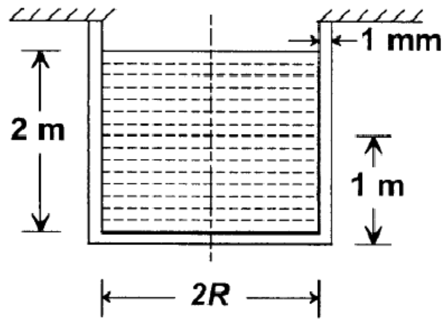
(a) $\dfrac{pd}{t}$ (b) $\dfrac{pd}{2t}$ (c) $\dfrac{pd}{4t}$ (d) $\dfrac{pd}{8t}$

GATE-2. Ans. (d) $\sigma_c = \dfrac{pd}{2t}$, $\sigma_l = \dfrac{pd}{4t}$, Maximum shear stress $= \dfrac{\sigma_c - \sigma_l}{2} = \dfrac{pd}{8t}$

Change in dimensions of a thin cylindrical shell due to an internal pressure

Statement for Linked Answers and Questions 3 and 4

A cylindrical container of radius R = 1 m, wall thickness 1 mm is filled with water up to a depth of 2 m and suspended along its upper rim. The density of water is 1000 kg/m³ and acceleration due to gravity is 10 m/s². The self-weight of the cylinder is negligible. The formula for hoop stress in a thin-walled cylinder can be used at all points along the height of the cylindrical container.



[GATE-2008]

GATE-3. The axial and circumferential stress (σ_a, σ_c) experienced by the cylinder wall at mid-depth (1 m as shown) are

(a) (10,10) MPa (b) (5,10) MPa (c) (10,5) MPa (d) (5,5)MPa

GATE-3. Ans. (a) Pressure (P) = h ρ g = $1\times 1000\times 10$ = 10 kPa

Axial Stress $(\sigma_a) \Rightarrow \sigma_a \times 2\pi Rt = \rho g \times \pi R^2 L$

or $\sigma_a = \dfrac{\rho g RL}{t} = \dfrac{1000\times 10\times 1\times 1}{1\times 10^{-3}} = 10\,\text{MPa}$

Circumferential Stress(σ_c)$= \dfrac{PR}{t} = \dfrac{10\times 1}{1\times 10^{-3}} = 10\,\text{MPa}$

GATE-4. If the Young's modulus and Poisson's ratio of the container material are 100 GPa and 0.3, respectively, the axial strain in the cylinder wall at mid-depth is:

(a) 2×10^{-5} (b) 6×10^{-5} (c) 7×10^{-5} (d) 1.2×10^{-5}

GATE-4. Ans. (c) $\varepsilon_a = \dfrac{\sigma_a}{E} - \mu\dfrac{\sigma_c}{E} = \dfrac{10}{100\times10^{-3}} - 0.3\times\dfrac{10}{100\times10^{-3}} = 7\times10^{-5}$

Previous 20-Years IES Questions

Circumferential or hoop stress

IES-1. Match List-I with List-II and select the correct answer: [IES-2002]

List-I (2-D Stress system loading)	List-II (Ratio of principal stresses)
A. Thin cylinder under internal pressure	**1.** 3.0
B. Thin sphere under internal pressure	**2.** 1.0
C. Shaft subjected to torsion	**3.** –1.0
	4. 2.0

Codes:	A	B	C		A	B	C
(a)	4	2	3	(b)	1	3	2
(c)	4	3	2	(d)	1	2	3

IES-1. Ans. (a)

IES-2. A thin cylinder of radius r and thickness t when subjected to an internal hydrostatic pressure P causes a radial displacement u, then the tangential strain caused is: [IES-2002]

(a) $\dfrac{du}{dr}$ (b) $\dfrac{1}{r}.\dfrac{du}{dr}$ (c) $\dfrac{u}{r}$ (d) $\dfrac{2u}{r}$

IES-2. Ans. (c)

IES-3. A thin cylindrical shell is subjected to internal pressure p. The Poisson's ratio of the material of the shell is 0.3. Due to internal pressure, the shell is subjected to circumferential strain and axial strain. The ratio of circumferential strain to axial strain is: [IES-2001]

(a) 0.425 (b) 2.25 (c) 0.225 (d) 4.25

IES-3. Ans. (d) Circumferential strain, $e_c = \dfrac{\sigma_c}{E} - \mu\dfrac{\sigma_l}{E} = \dfrac{pr}{2Et}(2-\mu)$

Longitudinal strain, $e_l = \dfrac{\sigma_l}{E} - \mu\dfrac{\sigma_c}{E} = \dfrac{pr}{2Et}(1-2\mu)$

IES-4. A thin cylindrical shell of diameter d, length 'l' and thickness t is subjected to an internal pressure p. What is the ratio of longitudinal strain to hoop strain in terms of Poisson's ratio (1/m)? [IES-2004]

(a) $\dfrac{m-2}{2m+1}$ (b) $\dfrac{m-2}{2m-1}$ (c) $\dfrac{2m-1}{m-2}$ (d) $\dfrac{2m+2}{m-1}$

IES-4. Ans. (b) longitudinal stress $(\sigma_l) = \dfrac{Pr}{2t}$

hoop stress $(\sigma_c) = \dfrac{Pr}{t}$

$$\therefore \frac{\epsilon_l}{\epsilon_c} = \frac{\dfrac{\sigma_l}{E} - \dfrac{1}{m}\dfrac{\sigma_c}{E}}{\dfrac{\sigma_c}{E} - \dfrac{1}{m}\dfrac{\sigma_l}{E}} = \frac{\dfrac{1}{2} - \dfrac{1}{m}}{1 - \dfrac{1}{2m}} = \frac{m-2}{2m-1}$$

IES-5. When a thin cylinder of diameter 'd' and thickness 't' is pressurized with an internal pressure of 'p', (1/m = μ is the Poisson's ratio and E is the modulus of elasticity), then [IES-1998]

(a) The circumferential strain will be equal to $\dfrac{pd}{2tE}\left(\dfrac{1}{2} - \dfrac{1}{m}\right)$

(b) The longitudinal strain will be equal to $\dfrac{pd}{2tE}\left(1 - \dfrac{1}{2m}\right)$

(c) The longitudinal stress will be equal to $\dfrac{pd}{2t}$

(d) The ratio of the longitudinal strain to circumferential strain will be equal to $\dfrac{m-2}{2m-1}$

IES-5. Ans. (d) Ratio of longitudinal strain to circumferential strain

$$= \frac{\sigma_l - \left(\dfrac{1}{m}\right)\sigma_c}{\sigma_c - \left(\dfrac{1}{m}\right)\sigma_l} = \frac{\sigma_l - \left(\dfrac{1}{m}\right)\{2\sigma_l\}}{\{2\sigma_l\} - \left(\dfrac{1}{m}\right)\sigma_l} = \frac{m-2}{2m-1}$$

Circumferential strain, $e_c = \dfrac{\sigma_c}{E} - \mu\dfrac{\sigma_l}{E} = \dfrac{pr}{2Et}(2-\mu)$

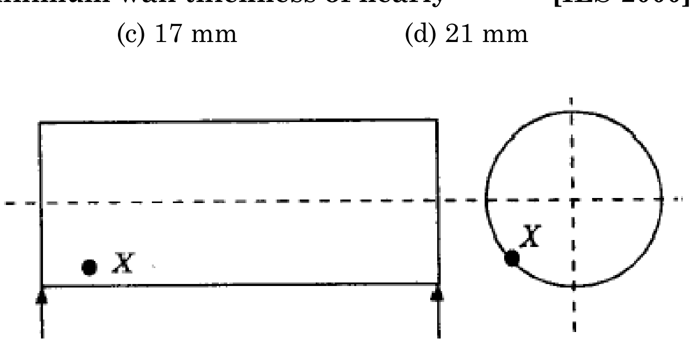
Longitudinal strain, $e_l = \dfrac{\sigma_l}{E} - \mu\dfrac{\sigma_c}{E} = \dfrac{pr}{2Et}(1-2\mu)$

IES-6. A thin cylinder contains fluid at a pressure of 500 N/m², the internal diameter of the shell is 0.6 m and the tensile stress in the material is to be limited to 9000 N/m². The shell must have a minimum wall thickness of nearly [IES-2000]

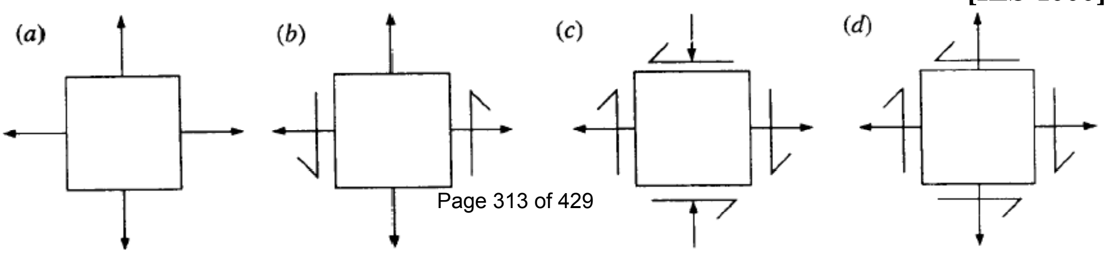
(a) 9 mm (b) 11 mm (c) 17 mm (d) 21 mm

IES-6. Ans. (c)

IES-7. A thin cylinder with closed lids is subjected to internal pressure and supported at the ends as shown in figure. The state of stress at point X is as represented in



[IES-1999]



IES-7. Ans. (a) Point 'X' is subjected to circumferential and longitudinal stress, i.e. tension on all faces, but there is no shear stress because vessel is supported freely outside.

IES-8. A thin cylinder with both ends closed is subjected to internal pressure p. The longitudinal stress at the surface has been calculated as σ_o. Maximum shear stress at the surface will be equal to: [IES-1999]

(a) $2\sigma_o$ (b) $1.5\sigma_o$ (c) σ_o (d) $0.5\sigma_o$

IES-8. Ans. (d)

Longitudinal stress $= \sigma_o$ *and* hoop stress $= 2\sigma_o$ Max. shear stress $= \dfrac{2\sigma_o - \sigma_o}{2} = \dfrac{\sigma_o}{2}$

IES-9. A metal pipe of 1m diameter contains a fluid having a pressure of 10 kgf/cm². lf the permissible tensile stress in the metal is 200 kgf/cm², then the thickness of the metal required for making the pipe would be: [IES-1993]

(a) 5 mm (b) 10 mm (c) 20 mm (d) 25 mm

IES-9. Ans. (d) Hoop stress $= \dfrac{pd}{2t}$ $or\ 200 = \dfrac{10\times 100}{2\times t}$ $or\ t = \dfrac{1000}{400} = 2.5\,cm$

IES-10. Circumferential stress in a cylindrical steel boiler shell under internal pressure is 80 MPa. Young's modulus of elasticity and Poisson's ratio are respectively 2×10^5 MPa and 0.28. The magnitude of circumferential strain in the boiler shell will be: [IES-1999]

(a) 3.44×10^{-4} (b) 3.84×10^{-4} (c) 4×10^{-4} (d) 4.56×10^{-4}

IES-10. Ans. (a) Circumferential strain = $\dfrac{1}{E}(\sigma_1 - \mu\sigma_2)$

Since circumferential stress $\sigma_1 =$ 80 MPa and longitudinal stress $\sigma_2 =$ 40 MPa

$$\therefore \text{Circumferential strain} = \frac{1}{2\times 10^5 \times 10^6}\left[80 - 0.28\times 40\right]\times 10^6 = 3.44\ \text{x}10^{-4}$$

IES-11. A penstock pipe of 10m diameter carries water under a pressure head of 100 m. If the wall thickness is 9 mm, what is the tensile stress in the pipe wall in MPa? [IES-2009]

(a) 2725 (b) 545·0 (c) 272·5 (d) 1090

IES-11. Ans. (b) Tensile stress in the pipe wall = Circumferential stress in pipe wall = $\dfrac{Pd}{2t}$

Where, $P = \rho gH = 980000 N/m^2$

$$\therefore \text{Tensile stress} = \frac{980000\times 10}{2\times 9\times 10^{-3}} = 544.44\times 10^6 N/m^2 = 544.44 MN/m^2 = 544.44 MPa$$

IES-12. A water main of 1 m diameter contains water at a pressure head of 100 metres. The permissible tensile stress in the material of the water main is 25 MPa. What is the minimum thickness of the water main? (Take g = 10 m/s^2). [IES-2009]

(a) 10 mm (b) 20mm (c) 50 mm (d) 60 mm

IES-12. Ans. (b) Pressure in the main $= \rho gh = 1000\times 10\times 1000 = 10^6\ N/mm^2 = 1000$ KPa

Hoop stress $= \sigma_c = \dfrac{Pd}{2t}$

$$\therefore \quad t = \frac{Pd}{2\sigma_c} = \frac{(10^6)(1)}{2\times 25\times 10^6} = \frac{1}{50} m = 20\ mm$$

Longitudinal stress

IES-13. Hoop stress and longitudinal stress in a boiler shell under internal pressure are 100 MN/m² and 50 MN/m² respectively. Young's modulus of elasticity and Poisson's ratio of the shell material are 200 GN/m² and 0.3 respectively. The hoop strain in boiler shell is: [IES-1995]

(a) 0.425×10^{-3} (b) 0.5×10^{-3} (c) 0.585×10^{-3} (d) 0.75×10^{-3}

IES-13. Ans. (a) $\text{Hoop strain} = \dfrac{1}{E}\left(\sigma_h - \mu\sigma_l\right) = \dfrac{1}{200\times1000}\left[100 - 0.3\times50\right] = 0.425\times10^{-3}$

IES-14. In strain gauge dynamometers, the use of how many active gauge makes the dynamometer more effective? [IES 2007]

(a) Four (b) Three (c) Two (d) One

IES-14. Ans. (b)

Volumetric strain

IES-15. Circumferential and longitudinal strains in a cylindrical boiler under internal steam pressure are ε_1 and ε_2 respectively. Change in volume of the boiler cylinder per unit volume will be: [IES-1993; IAS 2003]

(a) $\varepsilon_1 + 2\varepsilon_2$ (b) $\varepsilon_1\varepsilon_2^2$ (c) $2\varepsilon_1 + \varepsilon_2$ (d) $\varepsilon_1^2\varepsilon_2$

IES-15. Ans. (c) Volumetric stream = 2 × circumferential strain + longitudinal strain

IES-16. The volumetric strain in case of a thin cylindrical shell of diameter d, thickness t, subjected to internal pressure p is: [IES-2003; IAS 1997]

(a) $\dfrac{pd}{2tE}.\left(3-2\mu\right)$ (b) $\dfrac{pd}{3tE}.\left(4-3\mu\right)$ (c) $\dfrac{pd}{4tE}.\left(5-4\mu\right)$ (d) $\dfrac{pd}{4tE}.\left(4-5\mu\right)$

(Where E = Modulus of elasticity, μ = Poisson's ratio for the shell material)

IES-16. Ans. (c) Remember it.

Spherical Vessel

IES-17. For the same internal diameter, wall thickness, material and internal pressure, the ratio of maximum stress, induced in a thin cylindrical and in a thin spherical pressure vessel will be: [IES-2001]

(a) 2 (b) 1/2 (c) 4 (d) 1/4

IES-17. Ans. (a)

IES-18. From design point of view, spherical pressure vessels are preferred over cylindrical pressure vessels because they [IES-1997]

(a) Are cost effective in fabrication
(b) Have uniform higher circumferential stress
(c) Uniform lower circumferential stress
(d) Have a larger volume for the same quantity of material used

IES-18. Ans. (d)

Previous 20-Years IAS Questions

Circumferential or hoop stress

IAS-1. The ratio of circumferential stress to longitudinal stress in a thin cylinder subjected to internal hydrostatic pressure is: [IAS 1994]

(a) 1/2 (b) 1 (c) 2 (d) 4

IAS-1. Ans. (c)

IAS-2. A thin walled water pipe carries water under a pressure of 2 N/mm² and discharges water into a tank. Diameter of the pipe is 25 mm and thickness is 2·5 mm. What is the longitudinal stress induced in the pipe? [IAS-2007]
(a) 0 (b) 2 N/mm² (c) 5 N/mm² (d) 10 N/mm²

IAS-2. Ans. (c) $\sigma = \frac{\Pr}{2t} = \frac{2\times 12.5}{2\times 2.5} = 5\ \text{N/mm}^2$

IAS-3. A thin cylindrical shell of mean diameter 750 mm and wall thickness 10 mm has its ends rigidly closed by flat steel plates. The shell is subjected to internal fluid pressure of 10 N/mm² and an axial external pressure P₁. If the longitudinal stress in the shell is to be zero, what should be the approximate value of P₁? [IAS-2007]
(a) 8 N/mm² (b) 9 N/mm² (c) 10 N/mm² (d) 12 N/mm²

IAS-3. Ans. (c) Tensile longitudinal stress due to internal fluid pressure $(\delta_1)_t = \frac{10\times\left(\frac{\pi\times 750^2}{4}\right)}{\pi\times 750\times 10}$ tensile. Compressive longitudinal stress due to external pressure p_1 $(\delta_1)_c = \frac{P_1\times\left(\frac{\pi\times 750^2}{4}\right)}{\pi\times 750\times 10}$ compressive. For zero longitudinal stress $(\delta_1)_t = (\delta_1)_c$.

IAS-4. Assertion (A): A thin cylindrical shell is subjected to internal fluid pressure that induces a 2-D stress state in the material along the longitudinal and circumferential directions. [IAS-2000]
Reason(R): The circumferential stress in the thin cylindrical shell is two times the magnitude of longitudinal stress.
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-4. Ans. (b) For thin cell $\sigma_c = \frac{\Pr}{t}$ $\sigma_l = \frac{\Pr}{2t}$

IAS-5. Match List-I (Terms used in thin cylinder stress analysis) with List-II (Mathematical expressions) and select the correct answer using the codes given below the lists: [IAS-1998]

List-I	**List-II**
A. Hoop stress	**1.** pd/4t
B. Maximum shear stress	**2.** pd/2t
C. Longitudinal stress	**3.** pd/2σ
D. Cylinder thickness	**4.** pd/8t

Codes:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	2	3	1	4	(b)	2	3	4	1
(c)	2	4	3	1	(d)	2	4	1	3

IAS-5. Ans. (d)

Longitudinal stress

IAS-6. Assertion (A): For a thin cylinder under internal pressure, At least three strain gauges is needed to know the stress state completely at any point on the shell.

Reason (R): If the principal stresses directions are not know, the minimum number of strain gauges needed is three in a biaxial field. [IAS-2001]

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-6. Ans. (d) For thin cylinder, variation of radial strain is zero. So only circumferential and longitudinal strain has to measurer so only two strain gauges are needed.

Maximum shear stress

IAS-7. The maximum shear stress is induced in a thin-walled cylindrical shell having an internal diameter 'D' and thickness't' when subject to an internal pressure 'p' is equal to: [IAS-1996]

(a) pD/t (b) pD/2t (c) pD/4t (d) pD/8t

IAS-7. Ans. (d) Hoop stress$(\sigma_c) = \frac{PD}{2t}$ and Longitudinal stress$(\sigma_l) = \frac{PD}{4t}$ $\therefore \tau_{max} = \frac{\sigma_c - \sigma_l}{2} = \frac{PD}{8t}$

Volumetric strain

IAS-8. Circumferential and longitudinal strains in a cylindrical boiler under internal steam pressure are ε_1 and ε_2 respectively. Change in volume of the boiler cylinder per unit volume will be: [IES-1993; IAS 2003]

(a) $\varepsilon_1 + 2\varepsilon_2$ (b) $\varepsilon_1\varepsilon_2^2$ (c) $2\varepsilon_1 + \varepsilon_2$ (d) $\varepsilon_1^2\varepsilon_2$

IAS-8. Ans. (c) Volumetric stream = 2 x circumferential strain + longitudinal strain.

IAS-9. The volumetric strain in case of a thin cylindrical shell of diameter d, thickness t, subjected to internal pressure p is: [IES-2003; IAS 1997]

(a) $\frac{pd}{2tE}.(3-2\mu)$ (b) $\frac{pd}{3tE}.(4-3\mu)$ (c) $\frac{pd}{4tE}.(5-4\mu)$ (d) $\frac{pd}{4tE}.(4-5\mu)$

(Where E = Modulus of elasticity, μ = Poisson's ratio for the shell material)

IAS-9. Ans. (c) Remember it.

IAS-10. A thin cylinder of diameter 'd' and thickness 't' is subjected to an internal pressure 'p' the change in diameter is (where E is the modulus of elasticity and μ is the Poisson's ratio) [IAS-1998]

(a) $\frac{pd^2}{4tE}(2-\mu)$ (b) $\frac{pd^2}{2tE}(1+\mu)$ (c) $\frac{pd^2}{tE}(2+\mu)$ (d) $\frac{pd^2}{4tE}(2+\mu)$

IAS-10. Ans. (a)

IAS-11. The percentage change in volume of a thin cylinder under internal pressure having hoop stress = 200 MPa, E = 200 GPa and Poisson's ratio = 0·25 is: [IAS-2002]

(a) 0.40 (b) 0·30 (c) 0·25 (d) 0·20

IAS-11. Ans. (d) Hoop stress$\left(\sigma_t\right) = \frac{\Pr}{t} = 200\times10^6 P_a$

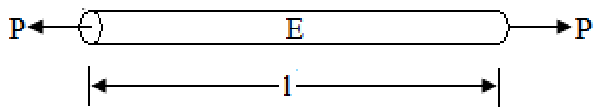
$$\text{Volumetric strain } (e_v) = \frac{\Pr}{2Et}(5-4\mu) = \frac{\sigma_t}{2E}(5-4\mu)$$

$$= \frac{200\times10^6}{2\times200\times10^9}(5-4\times0.25) = \frac{2}{1000}$$

IAS-12. A round bar of length *l*, elastic modulus E and Poisson's ratio μ is subjected to an axial pull 'P'. What would be the change in volume of the bar? [IAS-2007]

(a) $\frac{Pl}{(1-2\mu)E}$ (b) $\frac{Pl(1-2\mu)}{E}$ (c) $\frac{Pl\mu}{E}$ (d) $\frac{Pl}{\mu E}$

IAS-12. Ans. (b)



$$\sigma_x = \frac{P}{A}, \quad \sigma_y = 0 \quad \text{and} \quad \sigma_z = 0$$

$$\text{or } \varepsilon_x = \frac{\sigma_x}{E}, \ \varepsilon_y = -\mu\frac{\sigma_x}{E}$$

$$\text{and } \varepsilon_z = -\mu\frac{\sigma_x}{E}$$

$$\text{or } \varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x}{E}(1-2\mu) = \frac{P}{AE}(1-2\mu)$$

$$\delta V = \varepsilon_v \times V = \varepsilon_v.Al = \frac{Pl}{E}(1-2\mu)$$

IAS-13. If a block of material of length 25 cm. breadth 10 cm and height 5 cm undergoes a volumetric strain of 1/5000, then change in volume will be: [IAS-2000]

(a) 0.50 cm^3 (b) 0.25 cm^3 (c) 0.20 cm^3 (d) 0.75 cm^3

IAS-13. Ans. (b)

$$\text{Volumetric strain}(\varepsilon_v) = \frac{\text{Volume change}(\delta V)}{\text{Initial volume}(V)}$$

$$\text{or } (\delta V) = \varepsilon_v \times V = \frac{1}{5000}\times25\times10\times5 = 0.25cm^3$$

Previous Conventional Questions with Answers

Conventional Question GATE-1996

Question: **A thin cylinder of 100 mm internal diameter and 5 mm thickness is subjected to an internal pressure of 10 MPa and a torque of 2000 Nm. Calculate the magnitudes of the principal stresses.**

Answer: Given: d = 100 mm = 0.1 m; t = 5 mm = 0.005 m; D = d + 2t = 0.1 + 2 x 0.005 = 0.11 m p = 10 MPa, 10 x 10^6N/m^2; T= 2000 Nm.

Longitudinal stress, $\sigma_l = \sigma_x = \frac{pd}{4t} = \frac{10\times10^6\times0.1}{4\times0.005} = 50\times10^6 N/m^2 = 50MN/m^2$

Circumferential stress, $\sigma_c = \sigma_y = \frac{pd}{2t} = \frac{10\times10^6\times0.1}{2\times0.005} = 100MN/m^2$

To find the shear stress, using Torsional equation,

$\frac{T}{J} = \frac{\tau}{R}$, we have

$$\tau = \tau_{xy} = \frac{TR}{J} = \frac{T\times R}{\frac{\pi}{32}\left(D^4 - d^4\right)} = \frac{2000\times(0.05+0.005)}{\frac{\pi}{32}\left(0.11^4 - 0.1^4\right)} = 24.14MN/m^2$$

Principal stresses are:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

$$= \frac{50+100}{2} \pm \sqrt{\left(\frac{50-100}{2}\right)^2 + (24.14)^2}$$

$$= 75 \pm 34.75 = 109.75 \text{ and } 40.25MN/m^2$$

σ_1(Major principal stress) $= 109.75MN/m^2$;

σ_2(minor principal stress) $= 40.25MN/m^2$;

Conventional Question IES-2008

Question: **A thin cylindrical pressure vessel of inside radius 'r' and thickness of metal 't' is subject to an internal fluid pressure p. What are the values of**

(i) Maximum normal stress?

(ii) Maximum shear stress?

Answer: Circumferential (Hoop) stress $(\sigma_c) = \frac{p.r}{t}$

Longitudinal stress $(\sigma_\ell) == \frac{p.r}{2t}$

Therefore (ii) Maximum shear stress, (τ_{max}) $= \frac{\sigma_c - \sigma_\ell}{2} = \frac{p.r}{4t}$

Conventional Question IES-1996

Question: **A thin cylindrical vessel of internal diameter d and thickness t is closed at both ends is subjected to an internal pressure P. How much would be the hoop and longitudinal stress in the material?**

Answer: For thin cylinder we know that

Hoop or circumferential stress $(\sigma_c) = \frac{Pd}{2t}$

And longitudinal stress $(\sigma_\ell) = \frac{Pd}{4t}$

Therefore $\sigma_c = 2\sigma_\ell$

Conventional Question IES-2009

Q. **A cylindrical shell has the following dimensions:**
Length = 3 m
Inside diameter = 1 m
Thickness of metal = 10 mm
Internal pressure = 1.5 MPa
Calculate the change in dimensions of the shell and the maximum intensity of shear stress induced. Take E = 200 GPa and Poisson's ratio $v = 0.3$ **[15-Marks]**

Ans. We can consider this as a thin cylinder.

Hoop stresses, $\sigma_1 = \frac{pd}{2t}$

Longitudinal stresses, $\sigma_2 = \frac{pd}{4t}$

Shear stress $= \frac{\sigma_1 - \sigma_2}{2}$

$$= \frac{pd}{8t}$$

Hence from the given data

$$\sigma_1 = \frac{1.5\times10^6\times1}{2\times10\times10^{-3}} = 0.75\times10^8$$
$$= 75 \text{ MPa}$$

$$\sigma_2 = \frac{1.5\times10^6\times1}{4\times10\times10^{-3}} = 37.5\times10^6$$
$$= 37.5 \text{ MPa}$$

ε_1 Hoop strain

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - v\sigma_2)$$
$$= \frac{Pd}{4tE}(2 - v)$$
$$= \frac{1.5\times10^6\times1}{4\times10\times10^{-3}\times200\times10^9}(2-0.3)$$
$$= \frac{37.5\times10^6}{200\times10^9}(2-0.3)$$
$$= 0.31875\times10^{-3}$$
$$\frac{\Delta d}{d} = 0.3187\times10^{-3}$$

$\therefore$ change in diameter,

$$\Delta d = 1\times0.31875\times10^{-3} \text{ m}$$
$$= 0.31875 \text{ mm}$$

Logitudinal strain, ε_2

$$\epsilon_2 = \frac{pd}{4tE}(1-2v)$$

$$= \frac{37.5\times10^6}{200\times10^9}(1-2\times0.3)$$

$$= 7.5\times10^{-5}$$

$$\frac{\Delta l}{l} = 7.5\times10^{-5}$$

$$\text{or } \Delta l = 7.5\times10^{-5}\times3$$

$$= 2.25\times10^{-4}\text{ m} = 0.225\text{mm}$$

$\Rightarrow$ Change in length = 0.225 mm and maximum shear stress,

$$\sigma = \frac{pd}{8t} = \frac{1.5\times10^6\times1}{8\times10\times10^{-3}}$$
$$= 18.75\text{ MPa}$$

Conventional Question IES-1998

Question: **A thin cylinder with closed ends has an internal diameter of 50 mm and a wall thickness of 2.5 mm. It is subjected to an axial pull of 10 kN and a torque of 500 Nm while under an internal pressure of 6 MN/m²**

(i) Determine the principal stresses in the tube and the maximum shear stress.

(ii) Represent the stress configuration on a square element taken in the load direction with direction and magnitude indicated; (schematic).

Answer: Given: d = 50 mm = 0.05 m D = d + 2t = 50 + 2 x 2.5 = 55 mm = 0.055 m;

Axial pull, P = 10 kN; T= 500 Nm; p = 6MN/m²

(i) Principal stresses ($\sigma_{1,2}$) in the tube and the maximum shear stress (t_{max}):

$$\sigma_x = \frac{pd}{4t} + \frac{P}{\pi dt} = \frac{6\times10^6\times0.05}{4\times2.5\times10^{-3}} + \frac{10\times10^3}{\pi\times0.05\times2.5\times10^{-3}}$$
$$= 30\times10^6 + 25.5\times10^6 = 55.5\times10^6\text{N/m}^2$$

$$\sigma_y = \frac{pd}{2t} = \frac{6\times10^6\times0.05}{2\times2.5\times10^{-3}} = 60\times10^6$$

Principal stresses are:

$$\sigma_{1,2} = \left(\frac{\sigma_x+\sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right) + \tau_{xy}^2} \quad ---(1)$$

$$\text{Use Torsional equation, } \frac{T}{J} = \frac{\tau}{R} \qquad ---(i)$$

$$\text{where } J = \frac{\pi}{32}\left(D^4 - d^4\right) = \frac{\pi}{32}\left[(0.055)^4 - (0.05)^4\right] = 2.848\times10^{-7}\text{m}^4$$

$$(J = \text{polar moment of inertia})$$

Substituting the values in (i), we get

$$\frac{500}{2.848\times10^{-7}}=\frac{\tau}{(0.055/2)}$$

or $$\tau=\frac{500\times(0.055/2)}{2.848\times10^{-7}}=48.28\times10^{6}\text{N}/\text{m}^2$$

Now, substituting the various values in eqn. (i), we have

$$\sigma_{1,2}=\left(\frac{55.5\times10^6+60\times10^6}{2}\right)\pm\sqrt{\left(\frac{55.5\times10^6-60\times10^6}{2}\right)+\left(48.28\times10^6\right)^2}$$

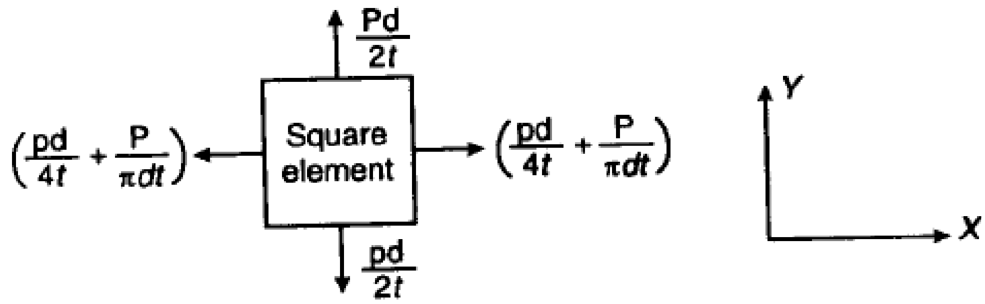
$$=\frac{(55.5+60)\times10^6}{2}\pm\sqrt{4.84\times10^{12}+2330.96\times10^{12}}$$

$$=57.75\times10^6\pm48.33\times10^6=106.08\text{MN}/\text{m}^2, 9.42\text{MN}/\text{m}^2$$

Principal stresses are : $\sigma_1=106.08\text{MN}/\text{m}^2$; $\sigma_2=9.42\text{MN}/\text{m}^2$

Maximum shear stress, $\tau_{max}=\dfrac{\sigma_1-\sigma_2}{2}=\dfrac{106.08-9.42}{2}=48.33\text{MN}/\text{m}^2$

(ii) Stress configuration on a square element:



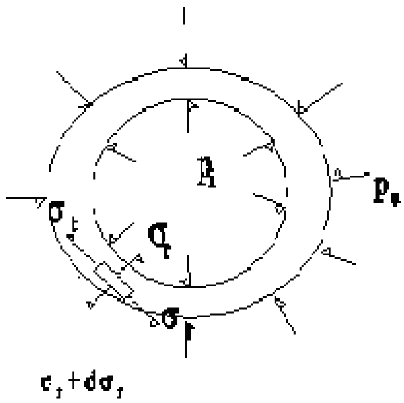
11. Thick Cylinder

Theory at a Glance (for IES, GATE, PSU)

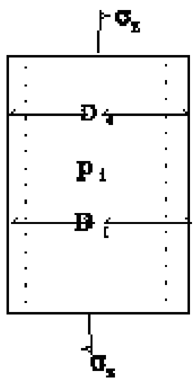
1. Thick cylinder

$$\frac{\text{Inner dia of the cylinder } (d_i)}{\text{wall thickness } (t)} < 15 \text{ or } 20$$

2. General Expression



(a)



(b)

3. Difference between the analysis of stresses in thin & thick cylinders

- In thin cylinders, it is assumed that the tangential stress σ_t is uniformly distributed over the cylinder wall thickness.

  In thick cylinder, the tangential stress σ_t has the highest magnitude at the inner surface of the cylinder & gradually decreases towards the outer surface.
- The radial stress σ_r is neglected in thin cylinders while it is of significant magnitude in case of thick cylinders.

4. Strain

- Radial strain, $\in_r = \frac{du}{dr}$.
- Circumferential /Tangential strain $\in_t = \frac{u}{r}$
- Axial strain, $\in_z = \frac{\sigma_z}{E} - \mu\left(\frac{\sigma_r}{E} + \frac{\sigma_t}{E}\right)$

5. Stress

- Axial stress, $\sigma_z = \dfrac{p_i r_i^2}{r_0^2 - r_i^2}$
- Radial stress, $\sigma_r = A - \dfrac{B}{r^2}$
- Circumferential /Tangential stress, $\sigma_t = A + \dfrac{B}{r^2}$

[Note: *Radial stress always compressive so its magnitude always –ive. But in some books they assume that compressive radial stress is positive and they use,* $\sigma_r = \dfrac{B}{r^2} - A$**]**

6. Boundary Conditions

At $r = r_i$, $\sigma_r = -p_i$

At $r = r_o$ $\sigma_r = -p_o$

7. $A = \dfrac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}$ and $B = (p_i - p_o)\dfrac{r_i^2 r_o^2}{(r_o^2 - r_i^2)}$

8. Cylinders with internal pressure (p_i) i.e. $p_o = 0$

- $\sigma_z = \dfrac{p_i r_i^2}{r_0^2 - r_i^2}$
- $\sigma_r = -\dfrac{p_i r_i^2}{r_0^2 - r_i^2}\left[\dfrac{r_0^2}{r^2} - 1\right]$ [-ive means compressive stress]
- $\sigma_t = +\dfrac{p_i r_i^2}{r_0^2 - r_i^2}\left[\dfrac{r_0^2}{r^2} + 1\right]$

(a) At the inner surface of the cylinder

(i) $r = r_i$

(ii) $\sigma_r = -p_i$

(iii) $\sigma_t = +\dfrac{p_i (r_o^2 + r_i^2)}{r_o^2 - r_i^2}$

(iv) $\tau_{max} = \dfrac{r_o^2}{r_o^2 - r_i^2} . p_i$

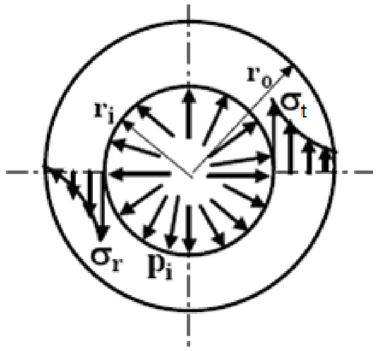
(b) At the outer surface of the cylinder

$(i)\ \mathrm{r} = \mathrm{r}_o$

$(ii)\ \sigma_r = 0$

$(iii)\ \sigma_t = \dfrac{2\mathrm{p_i r}_i^2}{r_o^2 - r_i^2}$

(c) Radial and circumferential stress distribution within the cylinder wall when only internal pressure acts.



9. Cylinders with External Pressure (p_o) i.e. $p_i = 0$

- $\sigma_r = -\dfrac{p_o r_o^2}{r_o^2 - r_i^2}\left[i - \dfrac{r_i^2}{r^2}\right]$
- $\sigma_t = -\dfrac{p_o r_o^2}{r_o^2 - r_i^2}\left[i + \dfrac{r_i^2}{r^2}\right]$

(a) At the inner surface of the cylinder

(i) $\mathrm{r} = r_i$

(ii) $\sigma_r = o$

(iii) $\sigma_t = -\dfrac{2p_o r_o^2}{r_o^2 - r_i^2}$

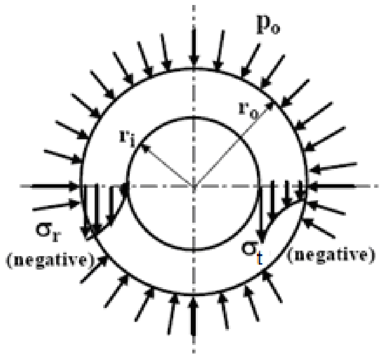
(b) At the outer surface of the cylinder

(i) $\mathrm{r} = \mathrm{r}_o$

(ii) $\sigma_r = -p_o$

(iii) $\sigma_t = -\dfrac{p_o(r_o^2 + r_i^2)}{r_o^2 - r_i^2}$

(c) Distribution of radial and circumferential stresses within the cylinder wall when only external pressure acts



10. Lame's Equation [for Brittle Material, open or closed end]

There is a no of equations for the design of thick cylinders. The choice of equation depends upon two parameters.

- Cylinder Material (Whether brittle or ductile)
- Condition of Cylinder ends (open or closed)

When the material of the cylinder is brittle, such as cast iron or cast steel, Lame's Equation is used to determine the wall thickness. Condition of cylinder ends may open or closed.

It is based on maximum principal stress theory of failure.

There principal stresses *at the inner surface of the cylinder* are as follows: (i) (ii) & (iii)

$$(i)\ \sigma_r = -p_i$$

$$(ii)\ \sigma_t = +\frac{p_i(r_0^2 + r_i^2)}{r_0^2 - r_i^2}$$

$$(iii)\ \sigma_z = +\frac{p_i r_i^2}{r_o^2 - r_i^2}$$

- $\sigma_t > \sigma_z > \sigma_r$
- σ_t is the criterion of design $\frac{r_o}{r_i} = \sqrt{\frac{\sigma_t + p_i}{\sigma_t - p_i}}$
- For $r_o = r_i + t$
- $\boxed{\mathrm{t} = r_i \times \left[\sqrt{\frac{\sigma_t + p_i}{\sigma_t - p_i}} - 1\right]}$ $(Lame's\ Equation)$
- $\sigma_t = \frac{\sigma_{ult}}{fos}$

11. Clavarino's Equation [for cylinders with closed end & made of ductile material]

When the material of a cylinder is ductile, such as mild steel or alloy steel, maximum strain theory of failure is used (St. Venant's theory) is used.

Three principal stresses at the *inner surface of the cylinder* are as follows (i) (ii) & (iii)

$$(i)\ \sigma_r = -p_i$$

$$(ii)\sigma_t = +\frac{p_i(r_o^2 + r_i^2)}{(r_o^2 - r_i^2)}$$

$$(iii)\sigma_z = +\frac{p_i r_i^2}{(r_o^2 - r_i^2)}$$

- $\in_t = \frac{1}{E}\left[\sigma_t - \left(\sigma_r + \sigma_z\right)\right]$
- $\in_t = \frac{\sigma}{E} = \frac{\sigma_{yld}/fos}{E}$
- Or $\sigma = \sigma_t - \mu(\sigma_r + \sigma_z)$. Where $\sigma = \frac{\sigma_{\text{yld}}}{\text{fos}}$
- σ is the criterion of design

$$\frac{r_o}{r_i} = \sqrt{\frac{\sigma + (1-2\mu)p_i}{\sigma - (1+\mu)p_i}}$$

- For $r_o = r_i + t$

$$t = r_i\left[\sqrt{\frac{\sigma + (1-2\mu)p_i}{\sigma - (1+\mu)p_i}} - 1\right] \quad \text{(Clavarion's Equation)}$$

12. Birne's Equation [for cylinders with open end & made of ductile material]

When the material of a cylinder is ductile, such as mild steel or alloy steel, maximum strain theory of failure is used (St. Venant's theory) is used.

Three principal stresses at the *inner surface of the cylinder* are as follows (i) (ii) & (iii)

$$(i)\ \sigma_r = -p_i$$

$$(ii)\sigma_t = +\frac{p_i(r_o^2 + r_i^2)}{(r_o^2 - r_i^2)}$$

$$(iii)\sigma_z = 0$$

- $\sigma = \sigma_t - \mu\sigma_r$ where $\sigma = \frac{\sigma_{\text{yld}}}{\text{fos}}$
- σ is the criterion of design

$$\frac{r_o}{r_i} = \sqrt{\frac{\sigma + (1-\mu)p_i}{\sigma - (1+\mu)p_i}}$$

- For $r_o = r_i + t$

$$t = r_i \times \left[\sqrt{\frac{\sigma + (1-\mu)p_i}{\sigma - (1+\mu)p_i}} - 1\right] \quad \text{(Birnie's Equation)}$$

13. Barlow's equation: [for high pressure gas pipe brittle or ductile material]

$$t = r_o \frac{p_i}{\sigma_t}$$ **[GAIL exam 2004]**

Where $\sigma_t = \dfrac{\sigma_y}{fos}$ *for ductile material*

$= \dfrac{\sigma_{ult}}{fos}$ *for brittle material*

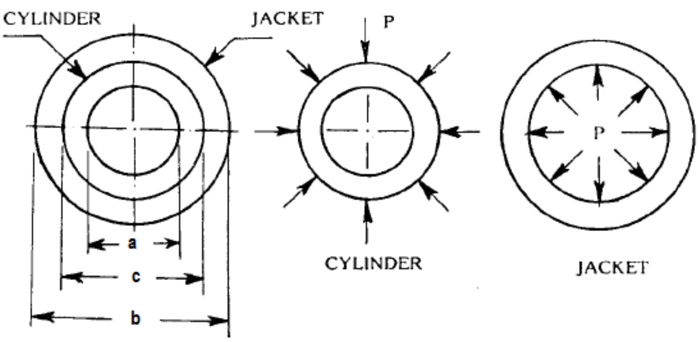
14. Compound Cylinder (A cylinder & A Jacket)

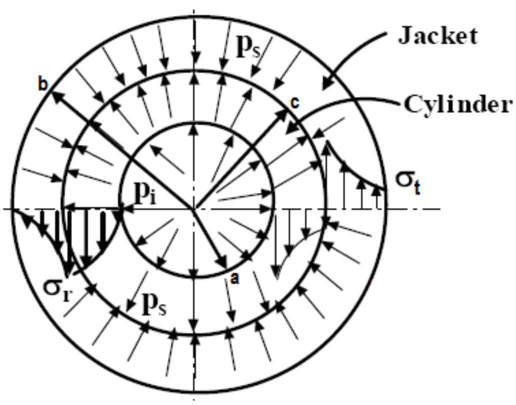
- When two cylindrical parts are assembled by shrinking or press-fitting, a contact pressure is created between the two parts. If the radii of the inner cylinder are a and c and that of the outer cylinder are (c- δ) and b, δ being the radial interference the contact pressure is given by:

$$P = \frac{E\delta}{c}\left[\frac{\left(b^2 - c^2\right)(c^2 - a^2)}{2c^2(b^2 - a^2)}\right]$$ Where E is the Young's modulus of the material

- The inner diameter of the jacket is slightly smaller than the outer diameter of cylinder
- When the jacket is heated, it expands sufficiently to move over the cylinder
- As the jacket cools, it tends to contract onto the inner cylinder, which induces residual compressive stress.
- There is a shrinkage pressure 'P' between the cylinder and the jacket.
- The pressure 'P' tends to contract the cylinder and expand the jacket
- The shrinkage pressure 'P' can be evaluated from the above equation for a given amount of interference δ
- The resultant stresses in a compound cylinder are found by supervision losing the 2- stresses
    - stresses due to shrink fit
    - stresses due to internal pressure

Derivation:





Due to interference let us assume δ_j = increase in inner diameter of jacket and δ_c = decrease in outer diameter of cylinder.

so $\delta = |\delta_j| + |\delta_c|$ i.e. without sign.

Now $\delta_j = \in_j c$ $\quad \left[\in_j = \text{tangential strain}\right]$

$$= \frac{1}{E}\left[\sigma_t - \mu\sigma_r\right]c$$

$$= \frac{cP}{E}\left[\frac{b^2+c^2}{b^2-c^2} + \mu\right] - - - (i) \qquad \begin{bmatrix} \sigma_t = \text{circumferential stress} \\ +\dfrac{p(b^2+c^2)}{(b^2-c^2)} \\ \sigma_r = -p\,(\text{radial stress}) \end{bmatrix}$$

And in similar way $\delta_c = \in_c c = \frac{1}{E}\left[\sigma_t - \mu\sigma_r\right]c \qquad \begin{bmatrix} \sigma_t = -\dfrac{p(c^2+a^2)}{(c^2-a^2)} \\ \sigma_r = -p \end{bmatrix}$

$$= -\frac{cP}{E}\left[\frac{c^2+a^2}{c^2-a^2} - \mu\right] - - - (ii)$$ Here -ive sign represents contraction

Adding (i) & (ii)

$$\therefore \delta = |\delta_j| + |\delta_c| = \frac{Pc}{E}\left[\frac{2c^2(b^2-a^2)}{(b^2-c^2)(c^2-a^2)}\right] \quad \text{or} \quad P = \frac{E\delta}{c}\left[\frac{(b^2-c^2)(c^2-a^2)}{2c^2(b^2-a^2)}\right]$$

15. Autofrettage

Autofrettage is a process of pre-stressing the cylinder before using it in operation.

We know that when the cylinder is subjected to internal pressure, the circumferential stress at the inner surface limits the pressure carrying capacity of the cylinder.

In autofrettage pre-stressing develops a residual compressive stresses at the inner surface. When the cylinder is actually loaded in operation, the residual compressive stresses at the inner surface begin to decrease, become zero and finally become tensile as the pressure is gradually increased. Thus autofrettage increases the pressure carrying capacity of the cylinder.

16. Rotating Disc

The radial & circumferential (tangential) stresses in a rotating disc of uniform thickness are given by

$$\sigma_r = \frac{\rho\omega^2}{8}(3+\mu)\left(R_0^2 + R_i^2 - \frac{R_0^2 R_i^2}{r^2} - r^2\right)$$

$$\sigma_t = \frac{\rho\omega^2}{8}(3+\mu)\left(R_0^2 + R_i^2 + \frac{R_0^2 R_i^2}{r^2} - \frac{1+3\mu}{3+\mu}.r^2\right)$$

Where R_i = Internal radius

R_o = External radius

ρ = Density of the disc material

ω = Angular speed

μ = Poisson's ratio.

Or, Hoop's stress, $\sigma_t = \left(\frac{3+\mu}{4}\right).\rho\omega^2.\left[R_0^2 + \left(\frac{1-\mu}{3+\mu}\right)R_i^{\,2}\right]$

Radial stress, $\sigma_r = \left(\frac{3+\mu}{8}\right).\rho\omega^2\left[R_0^2 - R_i^{\,2}\right]$

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Lame's theory

GATE-1. A thick cylinder is subjected to an internal pressure of 60 MPa. If the hoop stress on the outer surface is 150 MPa, then the hoop stress on the internal surface is: **[GATE-1996; IES-2001]**

(a) 105 MPa (b) 180 MPa (c) 210 MPa (d) 135 MPa

GATE-1. Ans. (c) If internal pressure = p_i; External pressure = zero

Circumferential or hoop stress (σ_c) = $\dfrac{p_i r_i^2}{r_o^2 - r_i^2}\left[\dfrac{r_o^2}{r^2}+1\right]$

At $p_i = 60\text{MPa}, \quad \sigma_c = 150\text{MPa} \text{ and } r = r_o$

$$\therefore 150 = 60\frac{r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r_o^2}+1\right] = 120\frac{r_i^2}{r_o^2 - r_i^2} \qquad \text{or } \frac{r_i^2}{r_o^2 - r_i^2} = \frac{150}{120} = \frac{5}{4} \qquad \text{or } \left(\frac{r_o}{r_i}\right)^2 = \frac{9}{5}$$

$\therefore$ at $r = r_i$

$$\sigma_c = 60\frac{r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r_i^2}+1\right] = 60\times\frac{5}{4}\times\left(\frac{9}{5}+1\right) = 210 \text{ MPa}$$

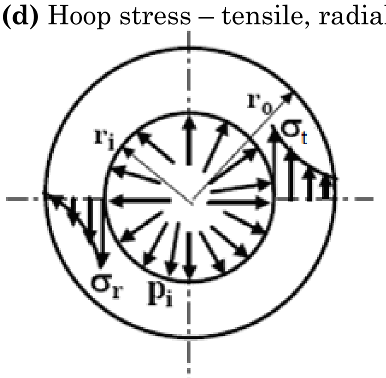
Previous 20-Years IES Questions

Thick cylinder

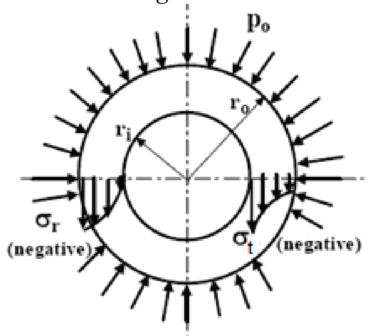
IES-1. If a thick cylindrical shell is subjected to internal pressure, then hoop stress, radial stress and longitudinal stress at a point in the thickness will be:

(a) Tensile, compressive and compressive respectively **[IES-1999]**
(b) All compressive
(c) All tensile
(d) Tensile, compressive and tensile respectively

IES-1. Ans. (d) Hoop stress – tensile, radial stress – compressive and longitudinal stress – tensile.



Radial and circumferential stress distribution within the cylinder wall when only internal pressure acts.



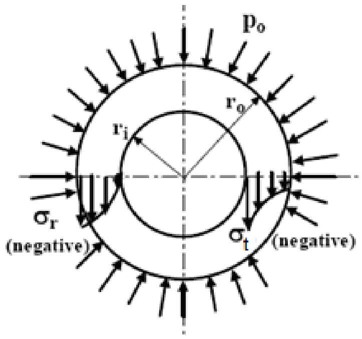
Distribution of radial and circumferential stresses within the cylinder wall when only external pressure acts.

IES-2. Where does the maximum hoop stress in a thick cylinder under external pressure occur? **[IES-2008]**

(a) At the outer surface (b) At the inner surface
(c) At the mid-thickness (d) At the 2/3rd outer radius

IES-2. Ans. (b)

Circumferential or hoop stress = σ_t

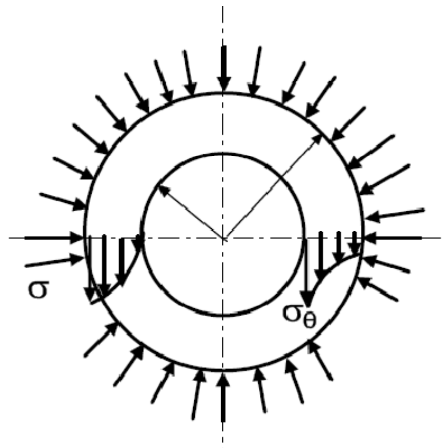


IES-3. In a thick cylinder pressurized from inside, the hoop stress is maximum at
(a) The centre of the wall thickness (b) The outer radius **[IES-1998]**
(c) The inner radius (d) Both the inner and the outer radii

IES-3. Ans. (c)

IES-4. Where does the maximum hoop stress in a thick cylinder under external pressure occur? [IES-2008]
(a) At the outer surface (b) At the inner surface
(c) At the mid-thickness (d) At the 2/3rd outer radius

IES-4. Ans. (a) Maximum hoop stress in thick cylinder under external pressure occur at the outer surface.



IES-5. A thick-walled hollow cylinder having outside and inside radii of 90 mm and 40 mm respectively is subjected to an external pressure of 800 MN/m^2. The maximum circumferential stress in the cylinder will occur at a radius of
[IES-1998]
(a) 40 mm (b) 60 mm (c) 65 mm (d) 90 mm

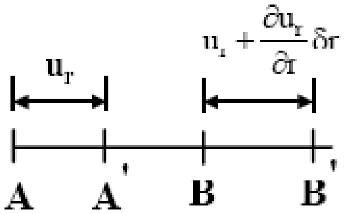
IES-5. Ans. (a)

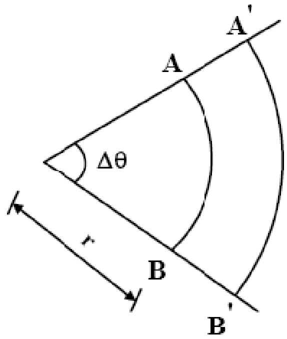
IES-6. In a thick cylinder, subjected to internal and external pressures, let r_1 and r_2 be the internal and external radii respectively. Let u be the radial displacement of a material element at radius r, $r_2 \geq r \geq r_1$. Identifying the cylinder axis as z axis, the radial strain component ε_{rr} is: [IES-1996]
(a) u/r (b) u/θ (c) du/dr (d) du/dθ

IES-6. Ans. (c) The strains ε_r and ε_θ may be given by

$$\varepsilon_r = \frac{\partial u_r}{\partial r} = \frac{1}{E}\left[\sigma_r - \upsilon\sigma_\theta\right] \qquad \text{since } \sigma_z = 0$$

$$\varepsilon_\theta = \frac{(r+u_r)\Delta\theta - r\Delta\theta}{r\Delta\theta} = \frac{u_r}{r} = \frac{1}{E}\left[\sigma_\theta - \upsilon\sigma_r\right]$$





Representation of radial and circumferential strain.

Lame's theory

IES-7. A thick cylinder is subjected to an internal pressure of 60 MPa. If the hoop stress on the outer surface is 150 MPa, then the hoop stress on the internal surface is: [GATE-1996; IES-2001]

(a) 105 MPa (b) 180 MPa (c) 210 MPa (d) 135 MPa

IES-7. Ans. (c) If internal pressure = p_i; External pressure = zero

Circumferential or hoop stress (σ_c) = $\frac{p_i r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r^2} + 1\right]$

At $p_i = 60\text{MPa}, \quad \sigma_c = 150\text{MPa} \text{ and } r = r_o$

$$\therefore 150 = 60\frac{r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r_o^2}+1\right] = 120\frac{r_i^2}{r_o^2 - r_i^2} \qquad \text{or } \frac{r_i^2}{r_o^2 - r_i^2} = \frac{150}{120} = \frac{5}{4} \qquad \text{or } \left(\frac{r_o}{r_i}\right)^2 = \frac{9}{5}$$

$\therefore$ at $r = r_i$

$$\sigma_c = 60\frac{r_i^2}{r_o^2 - r_i^2}\left[\frac{r_o^2}{r_i^2}+1\right] = 60 \times \frac{5}{4} \times \left(\frac{9}{5}+1\right) = 210 \text{ MPa}$$

IES-8. A hollow pressure vessel is subject to internal pressure. [IES-2005]
Consider the following statements:
1. **Radial stress at inner radius is always zero.**
2. **Radial stress at outer radius is always zero.**
3. **The tangential stress is always higher than other stresses.**
4. **The tangential stress is always lower than other stresses.**

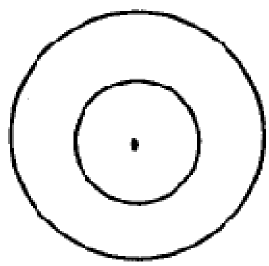
Which of the statements given above are correct?

(a) 1 and 3 (b) 1 and 4 (c) 2 and 3 (d) 2 and 4

IES-8. Ans. (c)

IES-9. A thick open ended cylinder as shown in the figure is made of a material with permissible normal and shear stresses 200 MPa and 100 MPa respectively. The ratio of permissible pressure based on the normal and shear stress is:
[d_i = 10 cm; d_o = 20 cm]

(a) 9/5 (b) 8/5
(c) 7/5 (d) 4/5



[IES-2002]

IES-9. Ans. (b)

Longitudinal and shear stress

IES-10. A thick cylinder of internal radius and external radius a and b is subjected to internal pressure p as well as external pressure p. Which one of the following statements is correct? [IES-2004]

The magnitude of circumferential stress developed is:

(a) Maximum at radius r = a (b) Maximum at radius r = b

(c) Maximum at radius r = $\sqrt{ab}$ (d) Constant

IES-10. Ans. (d)

$$\sigma_c = A + \frac{B}{r^2} \qquad A = \frac{P_i r_i^2 - P_o r_o^2}{r_o^2 - r_i^2} = \frac{Pa^2 - Pb^2}{b^2 - a^2} = -P$$

$$\therefore \sigma_c = -P \qquad B = \frac{(P_i - P_o) r_o^2 r_i^2}{r_o^2 - r_i^2} = o$$

IES-11. Consider the following statements: [IES-2007]
In a thick walled cylindrical pressure vessel subjected to internal pressure, the Tangential and radial stresses are:
1. **Minimum at outer side**
2. **Minimum at inner side**
3. **Maximum at inner side and both reduce to zero at outer wall**
4. **Maximum at inner wall but the radial stress reduces to zero at outer wall**

Which of the statements given above is/are correct?
(a) 1 and 2 (b) 1 and 3 (c) 1 and 4 (d) 4 only

IES-11. Ans. (c)

IES-12. Consider the following statements at given point in the case of thick cylinder subjected to fluid pressure: [IES-2006]
1. **Radial stress is compressive**
2. **Hoop stress is tensile**
3. **Hoop stress is compressive**
4. **Longitudinal stress is tensile and it varies along the length**
5. **Longitudinal stress is tensile and remains constant along the length of the cylinder**

Which of the statements given above are correct?
(a) Only 1, 2 and 4 (b) Only 3 and 4 (c) Only 1,2 and 5 (d) Only 1,3 and 5

IES-12. Ans. (c) 3. For internal fluid pressure Hoop or circumferential stress is **tensile.**
4. Longitudinal stress is tensile and remains **constant** along the length of the cylinder.

IES-13. A thick cylinder with internal diameter d and outside diameter 2d is subjected to internal pressure p. Then the maximum hoop stress developed in the cylinder is: [IES-2003]

(a) p (b) $\frac{2}{3}p$ (c) $\frac{5}{3}p$ (d) 2p

IES-13. Ans. (c) In thick cylinder, maximum hoop stress

$$\sigma_{hoop} = p \times \frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} = p \times \frac{d^2 + \left(\frac{d}{2}\right)^2}{d^2 - \left(\frac{d}{2}\right)^2} = \frac{5}{3}p$$

Compound or shrunk cylinder

IES-14. Autofrettage is a method of: [IES-1996; 2005; 2006]
(a) Joining thick cylinders (b) Relieving stresses from thick cylinders
(c) Pre-stressing thick cylinders (d) Increasing the life of thick cylinders

IES-14. Ans. (c)

IES-15. Match List-I with List-II and select the correct answer using the codes given below the Lists: [IES-2004]

List-I	**List-II**
A. Wire winding	**1.** Hydrostatic stress
B. Lame's theory	**2.** Strengthening of thin cylindrical shell
C. Solid sphere subjected to uniform pressure on the surface	**3.** Strengthening of thick cylindrical shell
D. Autofrettage	**4.** Thick cylinders

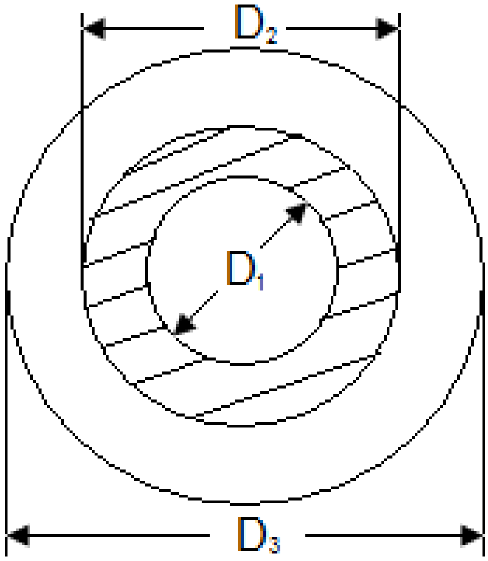
Coeds:	A	B	C	D		A	B	C	D
(a)	4	2	1	3	(b)	4	2	3	1
(c)	2	4	3	1	(d)	2	4	1	3

IES-15. Ans. (d)

IES-16. If the total radial interference between two cylinders forming a compound cylinder is δ and Young's modulus of the materials of the cylinders is E, then the interface pressure developed at the interface between two cylinders of the same material and same length is: [IES-2005]

(a) Directly proportional of E x δ (b) Inversely proportional of E/ δ
(c) Directly proportional of E/ δ (d) Inversely proportional of E / δ

IES-16. Ans. (a)



$$\delta = \frac{PD_2}{E}\left[\frac{2D_2^2\left(D_3^2 - D_1^2\right)}{\left[\left(D_3^2 - D_2^2\right)\left(D_2^2 - D_1^2\right)\right]}\right]$$

$\therefore P \alpha\ E.\delta$

Alternatively : if E ↑ then P ↑
and if δ ↑ then P ↑ so P α Eδ

IES-17. A compound cylinder with inner radius 5 cm and outer radius 7 cm is made by shrinking one cylinder on to the other cylinder. The junction radius is 6 cm and the junction pressure is 11 kgf/cm². The maximum hoop stress developed in the inner cylinder is: [IES-1994]

(a) 36 kgf/cm² compression (b) 36 kgf/cm² tension
(c) 72 kgf/cm² compression (d) 72 kgf/cm² tension.

IES-17. Ans. (c)

Thick Spherical Shell

IES-18. The hemispherical end of a pressure vessel is fastened to the cylindrical portion of the pressure vessel with the help of gasket, bolts and lock nuts. The bolts are subjected to: [IES-2003]

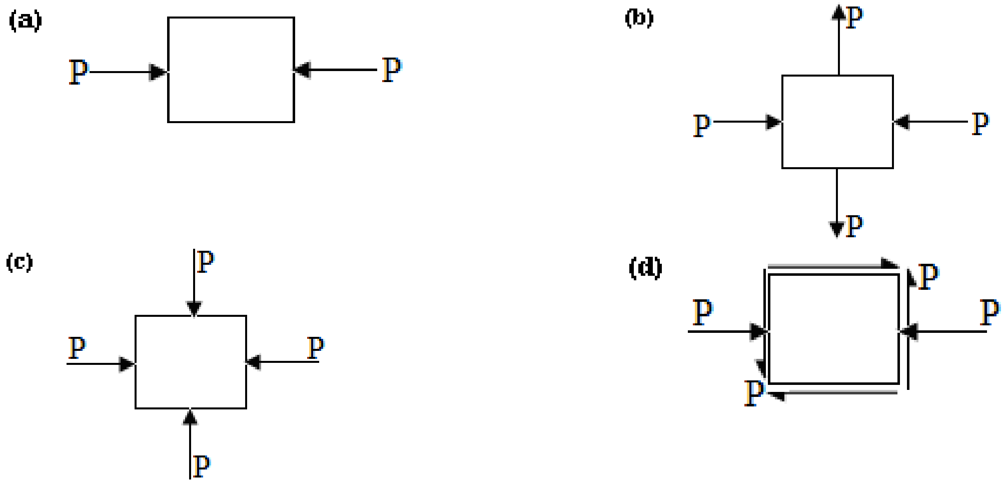
(a) Tensile stress (b) Compressive stress (c) Shear stress (d) Bearing stress

IES-18. Ans. (a)

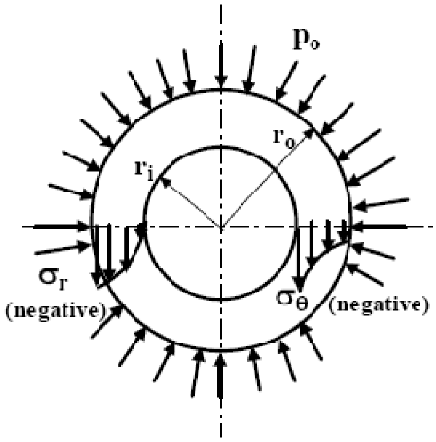
Previous 20-Years IAS Questions

Longitudinal and shear stress

IAS-1. A solid thick cylinder is subjected to an external hydrostatic pressure p. The state of stress in the material of the cylinder is represented as: [IAS-1995]



IAS-1. Ans. (c)



Distribution of radial and circumferential stresses within the cylinder wall when only external pressure acts.

Previous Conventional Questions with Answers

Conventional Question IES-1997

***Question*:** **The pressure within the cylinder of a hydraulic press is 9 MPa. The inside diameter of the cylinder is 25 mm. Determine the thickness of the cylinder wall, if the permissible tensile stress is 18 N/mm²**

***Answer*:** Given: P = 9 MPa = 9 N/mm², Inside radius, r_1 = 12.5 mm; σ_t = 18 N/mm²

Thickness of the cylinder:

$$\text{Using the equation; } \sigma_t = p\left[\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}\right], \text{we have}$$

$$18 = 9\left[\frac{r_2^2 + 12.5^2}{r_2^2 - 12.5^2}\right]$$

or $$r_2 = 21.65\text{mm}$$

$$\therefore \text{ Thickness of the cylinder} = r_2 - r_1 = 21.65 - 12.5 = 9.15\text{mm}$$

Conventional Question IES-2010

Q. **A spherical shell of 150 mm internal diameter has to withstand an internal pressure of 30 MN/m². Calculate the thickness of the shell if the allowable stress is 80 MN/m².**

Assume the stress distribution in the shell to follow the law

$$\sigma_r = a - \frac{2b}{r^3} \text{ and } \sigma_0 = a + \frac{b}{r^3}.$$ **[10 Marks]**

Ans. A spherical shell of 150 mm internal diameter internal pressure = 30 MPa.

Allowable stress = 80 MN/m²

Assume radial stress = $\sigma_r = a - \dfrac{2b}{r^3}$

Circumference stress = $\sigma_\theta = a + \dfrac{b}{r^3}$

At internal diameter (r)

$\sigma_r = -30\text{N}/\text{mm}^2$

$\sigma_\theta = 80\text{N}/\text{mm}^2$

$$-30 = a - \frac{2b}{(75)^3} \quad ....(i)$$

$$80 = a + \frac{b}{(75)^3} \quad ....(ii)$$

Soluing eq^n (i) & (ii)

$$b = \frac{110 \times 75^3}{3} \quad a = \frac{130}{3}$$

At outer Radius (R) radial stress should be zero

$$o = a - \frac{2b}{R^3}$$

$$R^3 = \frac{2b}{a} = \frac{2 \times 110 \times 75^3}{3 \times \frac{130}{3}} = 713942.3077$$

$$R = 89.376\text{mm}$$

There fore thickness of cylinder = $(R - r)$

$$= 89.376 - 75 = 14.376\,\text{mm}$$

Conventional Question IES-1993

Question: **A thick spherical vessel of inner 'radius 150 mm is subjected to an internal pressure of 80 MPa. Calculate its wall thickness based upon the**
(i) Maximum principal stress theory, and
(ii) Total strain energy theory.
Poisson's ratio = 0.30, yield strength = 300 MPa

Answer: Given:

$r_1 = 150\,\text{mm};\ p(\sigma_r) = 80\,\text{MPa} = 80 \times 10^6\,\text{N}/\text{m}^2;\ \mu = \frac{1}{m} = 0.30;$

$\sigma = 300\,\text{MPa} = 300 \times 10^6\,\text{N}/\text{m}^2$

Wall thickness t :

(i) Maximum principal stress theory :

We know that, $\sigma_r\left(\frac{K^2+1}{K^2-1}\right) \le \sigma \qquad \left(\text{Where } K = \frac{r_2}{r_1}\right)$

or $\quad 80 \times 10^6\left(\frac{K^2+1}{K^2-1}\right) \le 300 \times 10^6$

or $\quad K \ge 1.314$

or $\quad K = 1.314$

i.e. $\quad \frac{r_2}{r_1} = 1.314$ or $r_2 = r_1 \times 1.314 = 150 \times 1.314 = 197.1\text{mm}$

$\therefore$ Metal thickness, $t = r_2 - r_1 = 197.1 - 150 = 47.1$ mm

(ii) Total strain energy theory:

Use $\sigma_1^2 + \sigma_2^2 - \mu\sigma_1\sigma_2 \le \sigma_y^2$

$$\sigma^2 \geq \frac{2\sigma_r^2\left[K^4(1+\mu)+(1-\mu)\right]}{\left(K^2-1\right)^2}$$

$$\therefore \quad \left(300\times10^6\right)^2 \geq \frac{2\times\left(80\times10^6\right)^2\left[K^4(1+03)+(1-0.3)\right]}{\left(K^2-1\right)^2}$$

or $\quad 300^2\left(K^2-1\right)^2 = 2\times80^2\left(1.3K^4+0.7\right)$

gives $K = 1.86$ or 0.59

It is clear that $K > 1$

$\therefore K = 1.364$

or $\quad \frac{r_2}{r_1} = 1.364$ or $r_2 = 150\times1.364 = 204.6$ mm

$\therefore \quad t = r_2 - r_1 = 204.6 - 150 = 54.6$ mm

Conventional Question ESE-2002

***Question*:** **What is the difference in the analysis of think tubes compared to that for thin tubes? State the basic equations describing stress distribution in a thick tube.**

***Answer*:** The difference in the analysis of stresses in thin and thick cylinder:

(i) In thin cylinder, it is assumed that the tangential stress is uniformly distributed over the cylinder wall thickness. In thick cylinder, the tangential stress has highest magnitude at the inner surface of the cylinder and gradually decreases towards the outer surface.

(ii) The radial stress is neglected in thin cylinders, while it is of significant magnitude in case of thick cylinders.

Basic equation for describing stress distribution in thick tube is Lame's equation.

$$\sigma_r = \frac{B}{r^2} - A \quad \text{and} \quad \sigma_t = \frac{B}{r^2} + A$$

Conventional Question ESE-2006

***Question*:** **What is auto frettage?**
How does it help in increasing the pressure carrying capacity of a thick cylinder?

***Answer*:** Autofrettage is a process of pre-stressing the cylinder before using it in operation.

We know that when the cylinder is subjected to internal pressure, the circumferential stress at the inner surface limits the pressure carrying capacity of the cylinder.

In autofrettage pre-stressing develops a residual compressive stresses at the inner surface. When the cylinder is actually loaded in operation, the residual compressive stresses at the inner surface begin to decrease, become zero and finally become tensile as the pressure is gradually increased. Thus autofrettage increases the pressure carrying capacity of the cylinder.

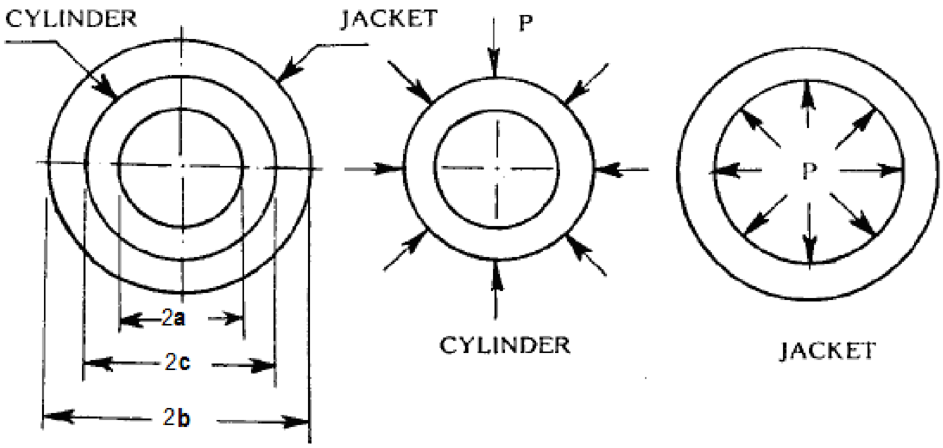
Conventional Question ESE-2001

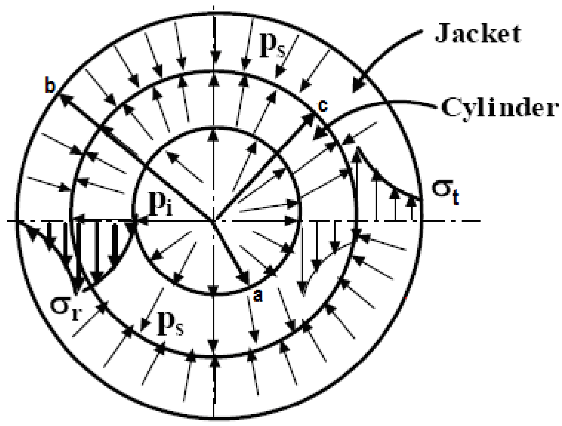
***Question*:** **When two cylindrical parts are assembled by shrinking or press-fitting, a contact pressure is created between the two parts. If the radii of the inner cylinder are a and c and that of the outer cylinder are (c- δ) and b, δ being the radial interference the contact pressure is given by:**

$$P = \frac{E\delta}{c}\left[\frac{\left(b^2-c^2\right)(c^2-a^2)}{2c^2(b^2-a^2)}\right]$$

Where E is the Young's modulus of the material, Can you outline the steps involved in developing this important design equation?

Answer:





Due to interference let us assume $\delta_j =$ increase in inner diameter of jacket and $\delta_c =$ decrease in outer diameter of cylinder.

so $\delta = |\delta_j| + |\delta_c|$ i.e. without sign.

Now $\delta_j = \in_j c$ $\quad [\in_j =$ tangential strain$]$

$$= \frac{1}{E}[\sigma_t - \mu\sigma_r]c$$

$$= \frac{cP}{E}\left[\frac{b^2+c^2}{b^2-c^2} + \mu\right] - - - (i) \qquad \left[\begin{array}{l}\sigma_t = \text{circumferential stress} \\ \quad + \dfrac{p(b^2+c^2)}{(b^2-c^2)} \\ \sigma_r = -p\,(\text{radial stress})\end{array}\right]$$

And in similar way $\delta_c = \in_c c$

$$= \frac{1}{E}[\sigma_t - \mu\sigma_r]c \qquad \left[\begin{array}{l}\sigma_t = -\dfrac{p(c^2+a^2)}{(c^2-a^2)} \\ \sigma_r = -p\end{array}\right]$$

$$= -\frac{cP}{E}\left[\frac{c^2+a^2}{c^2-a^2} - \mu\right] - - - (ii)$$ Here -ive sign represents contraction

Adding (i) & (ii)

$$\therefore \delta = |\delta_j| + |\delta_c| = \frac{Pc}{E}\left[\frac{2c^2(b^2-a^2)}{(b^2-c^2)(c^2-a^2)}\right]$$

$$\text{or } P = \frac{E\delta}{c}\left[\frac{(b^2-c^2)(c^2-a^2)}{2c^2(b^2-a^2)}\right]$$ Proved

Conventional Question ESE-2003

Question: **A steel rod of diameter 50 mm is forced into a bronze casing of outside diameter 90 mm, producing a tensile hoop stress of 30 MPa at the outside diameter of the casing.**

Find (i) The radial pressure between the rod and the casing

(ii) The shrinkage allowance and

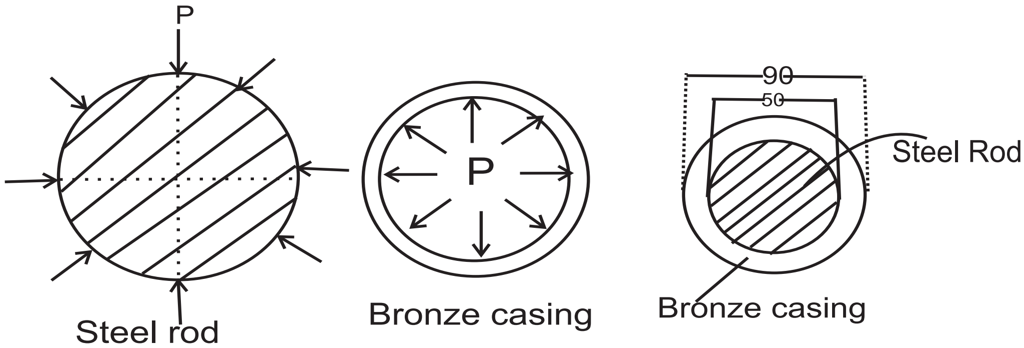
(iii) The rise in temperature which would just eliminate the force fit.

Assume the following material properties:

E_s = 2×10^5 MPa, $\mu_S = 0.25$ **,** $\alpha_s = 1.2\times10^{-5}/^oC$

E_b = 1×10^5 MPa , $\mu_b = 0.3,\ \ \mu_b = 1.9\times10^{-5}/^oC$

***Answer*:**



There is a shrinkage pressure P between the steel rod and the bronze casing. The pressure P tends to contract the steel rod and expand the bronze casing.

(i) Consider Bronze casing, According to Lames theory

$$\sigma_t = \frac{B}{r^2} + A \quad \text{Where A} = \frac{P_i r_i^2 - P_0 r_0^2}{r_0^2 - r_i^2}$$

$$\text{and B} = \frac{(P_i - P_0) r_0^2 r_i^2}{r_0^2 - r_i^2}$$

$P_i = P,\ \ P_0 = 0$ and

$$A = \frac{Pr_i^2}{r_0^2 - r_i^2}\ ,\ B = \frac{Pr_0^2 r_i^2}{r_0^2 - r_i^2} = \frac{2Pr_i^2}{r_0^2 - r_i^2}$$

$$\therefore 30 = \frac{B}{r_o^2} + A = \frac{Pr_i^2}{r_0^2 - r_i^2} + \frac{Pr_i^2}{r_0^2 - r_i^2} = \frac{2Pr_i^2}{r_0^2 - r_i^2}$$

$$or,\ P = \frac{30(r_0^2 - r_i^2)}{2r_i^2} = 15\left[\frac{r_0^2}{r_i^2} - 1\right] = 15\left[\left(\frac{90}{50}\right)^2 - 1\right]\text{MPa} = 33.6\text{MPa}$$

Therefore the radial pressure between the rod and the casing is P= 33.6 MPa.

(ii) The shrinkage allowance:

Let δ_j = increase in inert diameter of bronze casing

δ_C= decrease in outer diameter of steel rod

1st consider bronze casing:

$$\text{Tangential stress at the inner surface}(\sigma_t)_j = \frac{B}{r_i^2} + A$$

$$= \frac{Pr_0^2}{r_0^2 - r_i^2} + \frac{Pr_i^2}{r_0^2 - r_i^2} = \frac{P(r_0^2 + r_1^2)}{\left(r_0^2 - r_i^2\right)} = 33.6\times\left[\frac{\left(\frac{90}{50}\right)^2 + 1}{\left(\frac{90}{50}\right)^2 - 1}\right] = 63.6\text{MPa}$$

and radial stress$(\sigma_r)_j = -P = -33.6MPa$

longitudial stress$(\sigma_\ell)_j = 0$

Therefore tangential strain $(\varepsilon_t)_j = \frac{1}{E}\left[(\sigma_t)_j - \mu(\sigma_r)_j\right]$

$$= \frac{1}{1\times10^5}[63.6 + 0.3\times33.6] = 7.368\times10^{-4}$$

$$\therefore \delta_j = (\varepsilon_t)_j \times d_i = 7.368\times10^{-4}\times0.050 = 0.03684\,\text{mm}$$

2nd Consider steel rod:

Circumferential stress $(\sigma_t)_s = -P$

and radial stress $(\sigma_r)_s = -P$

$$\therefore \delta_c = (\in_t)_s \times d_i = \frac{1}{E_s}\left[(\sigma_t)_S - \mu(\sigma_r)_s\right]\times d_i$$

$$= -\frac{Pd_i}{E_s}(1-\mu) = -\frac{33.6\times0.050}{2\times10^5}\left[1-0.3\right] = -0.00588\,\text{mm [reduction]}$$

Total shrinkage $=\left|\delta_j\right| + \left|\delta_c\right| = 0.04272\,\text{mm}$ [it is diametral] = $0.02136\,\text{mm}$ [radial]

(iii) Let us temperature rise is (Δt)

As $\alpha_b > \alpha_s$ due to same temperature rise steel not will expand less than bronze casing. When their difference of expansion will be equal to the shrinkage then force fit will eliminate.

$$d_i\times\alpha_b\times\Delta t - d_i\times\alpha_s\times\Delta t = 0.04272$$

$$or\,\Delta t = \frac{0.04272}{d_i\left[\alpha_b-\alpha_s\right]} = \frac{0.04272}{50\times\left[1.9\times10^{-5}-1.2\times10^{-5}\right]} = 122^oC$$

Conventional Question AMIE-1998

Question: **A thick walled closed-end cylinder is made of an AI-alloy (E = 72 GPa, $\frac{1}{m} = 0.33$), has inside diameter of 200 mm and outside diameter of 800 mm. The cylinder is subjected to internal fluid pressure of 150 MPa. Determine the principal stresses and maximum shear stress at a point on the inside surface of the cylinder. Also determine the increase in inside diameter due to fluid pressure.**

Answer: Given: $r_1 = \frac{200}{2} = 100\text{mm} = 0.1\text{m}; r_2 = \frac{800}{2} = 400\text{mm} = 0.4; p = 150\text{MPa} = 150\text{MN/m}^2;$

$E = 72\text{GPa} = 72\times10^9\text{N/m}^2;\ \frac{1}{m} = 0.33 = \mu$

Principal stress and maximum shear stress:

Using the condition in Lame's equation:

$$\sigma_r = \frac{b}{r^2} - a$$

At $r = 0.1\text{m},\ \sigma_2 = +p = 150\text{MN/m}^2$

$r = 0.4\text{m},\ \sigma_2 = 0$

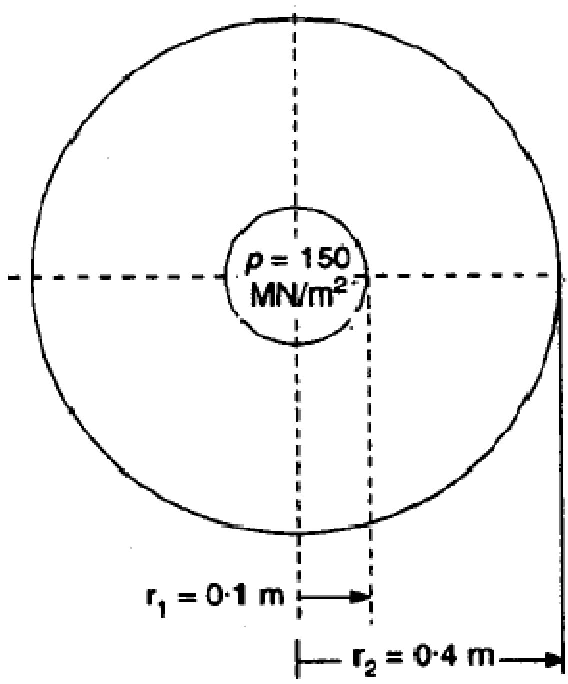
Substituting the values in the above equation we have

$$150 = \frac{b}{(0.1)^2} - a \qquad ----(i)$$

$$0 = \frac{b}{(0.4)^2} - a \qquad ----(ii)$$

From (i) and (ii), we get

$a = 10$ and $b = 1.6$



The circumferential (or hoop) stress by Lame's equation, is given by

$$\sigma_c = \frac{b}{r^2} + a$$

$\therefore (\sigma_c)_{max}$, at $r(=r_1) = 0.1\text{m} = \frac{1.6}{0.1^2} + 10 = 170\text{MN}/\text{m}^2$ (tensile), and

$(\sigma_c)_{min}$, at $r(=r_2) = 0.4\,\text{m} = \frac{1.6}{0.4^2} + 10 = 20\text{MN}/\text{m}^2$ (tensile).

$\therefore$ Principal stresses are 170 MN/m^2 and 20 MN/m^2

Maximum shear stress, $\tau_{max} = \frac{(\sigma_c)_{max} - (\sigma_c)_{min}}{2} = \frac{170 - 20}{2} = 75\text{MN}/\text{m}^2$

Increase in inside diameter, δd_1 :

We know, longitudinal (or axial) stress, $\sigma_l = \frac{pr_1^2}{r_2^2 - r_1^2} = \frac{150 \times (0.1)^2}{(0.4)^2 - (0.1)^2} = 10\,\text{MN}/\text{m}^2$

Circumferential (or hoop) strain at the inner radius, is given by :

$$\in_1 = \frac{1}{E}\left[\sigma_c + \mu(\sigma_r - \sigma_l)\right] = \frac{1}{72 \times 10^9}\left[170 \times 10^6 + 0.33(150 - 10) \times 10^6\right] = 0.003$$

Also, $\quad \in_1 = \frac{\delta d_1}{d_1}$

or $\quad 0.003 = \frac{\delta d_1}{0.1}$

$\delta d_1 = 0.003 \times 0.1 = 0.003\,\text{m}$ or 0.3 mm

12. Spring

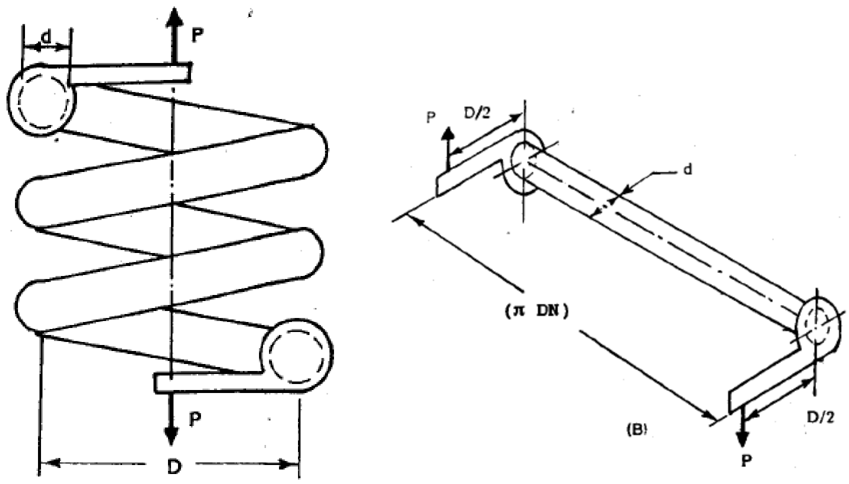
Theory at a Glance (for IES, GATE, PSU)

1. A spring is a mechanical device which is used for the efficient storage and release of energy.

2. Helical spring – stress equation

Let us a close-coiled helical spring has coil diameter D, wire diameter d and number of turn n. The spring material has a shearing modulus G. The spring index, $C = \frac{D}{d}$. If a force 'P' is exerted in both ends as shown.

The work done by the axial force 'P' is converted into strain energy and stored in the spring.



$$U = (\text{average torque}) \times (\text{angular displacement}) = \frac{T}{2} \times \theta$$

From the figure we get, $\theta = \frac{TL}{GJ}$

Torque $(T) = \frac{PD}{2}$

length of wire $(L) = \pi D n$

Polar moment of Inertia$(J) = \frac{\pi d^4}{32}$

Therefore $U = \frac{4P^2D^3n}{Gd^4}$

According to Castigliano's theorem, the displacement corresponding to force P is obtained by partially differentiating strain energy with respect to that force.

$$\text{Therefore } \delta = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left[\frac{4p^2D^3n}{Gd^4}\right] = \frac{8PD^3n}{Gd^4}$$

Axial deflection $\delta = \frac{8PD^3n}{Gd^4}$

Spring stiffness or spring constant $(k) = \frac{P}{\delta} = \frac{Gd^4}{8D^3n}$

The torsional shear stress in the bar, $\tau_1 = \dfrac{16T}{\pi d^3} = \dfrac{16(PD/2)}{\pi d^3} = \dfrac{8PD}{\pi d^3}$

The direct shear stress in the bar, $\tau_2 = \dfrac{P}{\left(\dfrac{\pi d^2}{4}\right)} = \dfrac{4P}{\pi d^2} = \dfrac{8PD}{\pi d^3}\left(\dfrac{0.5d}{D}\right)$

Therefore the total shear stress, $\tau = \tau_1 + \tau_2 = \dfrac{8PD}{\pi d^3}\left(1 + \dfrac{0.5d}{D}\right) = K_s \dfrac{8PD}{\pi d^3}$

$$\tau = K_s \frac{8PD}{\pi d^3}$$

Where $K_s = 1 + \dfrac{0.5d}{D}$ is correction factor for direct shear stress.

3. Wahl's stress correction factor

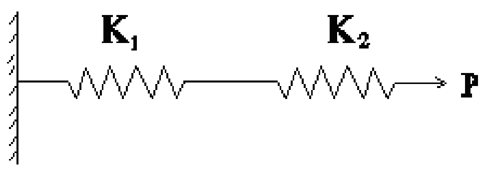
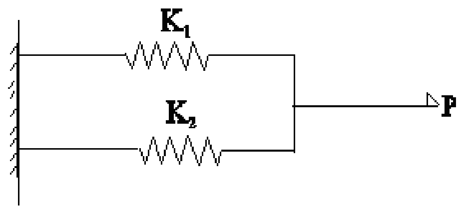
$$\tau = K \frac{8PD}{\pi d^3}$$

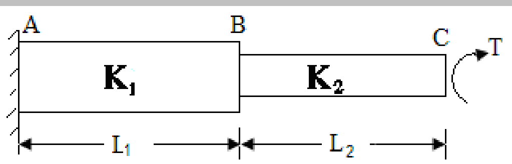
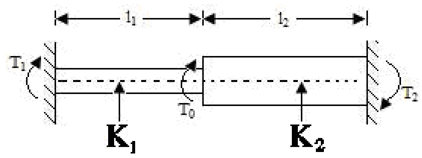
Where $K = \left(\dfrac{4C-1}{4C-4} + \dfrac{0.615}{C}\right)$ is known as Wahl's stress correction factor

Here K = K_sK_c; Where K_s is correction factor for direct shear stress and K_c is correction factor for stress concentration due to curvature.

Note: When the spring is subjected to a static force, the effect of stress concentration is neglected due to localized yielding. So we will use, $\tau = K_s \dfrac{8PD}{\pi d^3}$

4. Equivalent stiffness (k_{eq})

Spring in series $(\delta_e = \delta_1 + \delta_2)$	**Spring in Parallel** $(\delta_e = \delta_1 = \delta_2)$
	
$\dfrac{1}{K_{eq}} = \dfrac{1}{K_1} + \dfrac{1}{K_2}$ or $K_{eq} = \dfrac{K_1K_2}{K_1 + K_2}$	$K_{eq} = K_1 + K_2$

Shaft in series $(\theta = \theta_1 + \theta_2)$	**Shaft in Parallel** $(\theta_{eq} = \theta_1 = \theta_2)$
	

$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}$ or $K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$	$K_{eq} = K_1 + K_2$

5. Important note

- If a spring is cut into 'n' equal lengths then spring constant of each new spring = **nk**
- When a closed coiled spring is subjected to an axial couple M then the rotation, $\phi = \frac{64 M D n_c}{E d^4}$

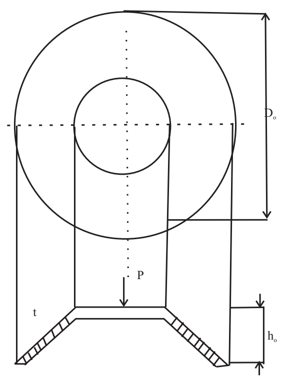
6. Laminated Leaf or Carriage Springs

- Central deflection, $\delta = \frac{3PL^3}{8Enbt^3}$
- Maximum bending stress, $\sigma_{max} = \frac{3PL}{2nbt^2}$

Where P = load on spring
b = width of each plate
n = no of plates
L= total length between 2 points
t =thickness of one plate.

7. Belleville Springs

$$\text{Load}, P = \frac{4E\delta}{(1-\mu^2)k_f D_0^2}\left[(h-\delta)\left(h-\frac{\delta}{2}\right)t + t^3\right]$$



Where, E = Modulus of elasticity
δ = Linear deflection
μ =Poisson's Ratio
k_f =factor for Belleville spring
D_o = outside diamerer
h = Deflection required to flatten Belleville spring
t = thickness

Note:

- **Total stiffness** of the springs k_{ror} = stiffness per spring × No of springs
- **In a leaf spring** ratio of stress between full length and graduated leaves = 1.5
- **Conical spring-** For application requiring variable stiffness
- **Belleville Springs** -For application requiring high capacity springs into small space

Objective Questions (GATE, IES, IAS)

Previous 20-Years GATE Questions

Helical spring

GATE-1. If the wire diameter of a closed coil helical spring subjected to compressive load is increased from 1 cm to 2 cm, other parameters remaining same, then deflection will decrease by a factor of: [GATE-2002]
(a) 16 (b) 8 (c) 4 (d) 2

GATE-1. Ans. (a) $\delta = \dfrac{8PD^3N}{G.d^4}$

GATE-2. A compression spring is made of music wire of 2 mm diameter having a shear strength and shear modulus of 800 MPa and 80 GPa respectively. The mean coil diameter is 20 mm, free length is 40 mm and the number of active coils is 10. If the mean coil diameter is reduced to 10 mm, the stiffness of the spring is approximately [GATE-2008]
(a) Decreased by 8 times (b) Decreased by 2 times
(c) Increased by 2 times (d) Increased by 8 times

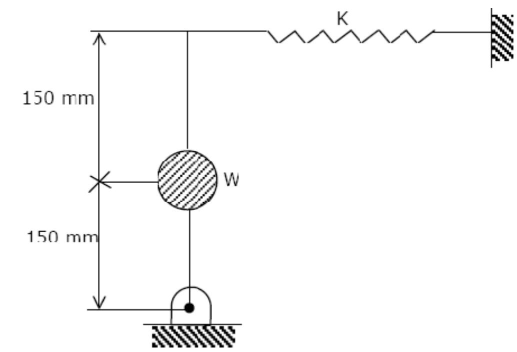
GATE-2. Ans. (d) Spring constant (K) $= \dfrac{P}{\delta} = \dfrac{G.d^4}{8D^3N}$ or $K \propto \dfrac{1}{D^3}$

$$\frac{K_2}{K_1} = \left(\frac{D_1}{D_2}\right)^3 = \left(\frac{20}{10}\right)^3 = 8$$

GATE-3. Two helical tensile springs of the same material and also having identical mean coil diameter and weight, have wire diameters d and d/2. The ratio of their stiffness is: [GATE-2001]
(a) 1 (b) 4 (c) 64 (d) 128

GATE-3. Ans. (c) Spring constant (K) $= \dfrac{P}{\delta} = \dfrac{G.d^4}{8D^3N}$ Therefore $k \propto \dfrac{d^4}{n}$

GATE-4. A uniform stiff rod of length 300 mm and having a weight of 300 N is pivoted at one end and connected to a spring at the other end. For keeping the rod vertical in a stable position the minimum value of spring constant K needed is:
(a) 300 N/m (b) 400N/m
(c) 500N/m (d) 1000 N/m



[GATE-2004]

GATE-4. Ans. (c) Inclined it to a very low angle, $d\theta$

For equilibrium taking moment about 'hinge'

$$W \times \left(\frac{l}{2}d\theta\right) - k\left(ld\theta\right) \times l = 0 \ \text{ or } k = \frac{W}{2l} = \frac{300}{2 \times 0.3} = 500\,N/m$$

GATE-5. A weighing machine consists of a 2 kg pan resting on spring. In this condition, with the pan resting on the spring, the length of the spring is 200 mm. When a mass of 20 kg is placed on the pan, the length of the spring becomes 100 mm. For the spring, the un-deformed length l_o and the spring constant k (stiffness) are: [GATE-2005]

(a) l_o = 220 mm, k = 1862 N/m (b) l_o = 210 mm, k = 1960 N/m
(c) l_o = 200 mm, k = 1960 N/m (d) l_o = 200 mm, k = 2156 N/m

GATE-5. Ans. (b) Initial length = l_o m and stiffness = k N/m

$$2 \times g = k(l_o - 0.2)$$

$$2 \times g + 20 \times g = k(l_o - 0.1)$$

Just solve the above equations.

Springs in Series

GATE-6. The deflection of a spring with 20 active turns under a load of 1000 N is 10 mm. The spring is made into two pieces each of 10 active coils and placed in parallel under the same load. The deflection of this system is: [GATE-1995]

(a) 20 mm (b) 10 mm (c) 5 mm (d) 2.5 mm

GATE-6. Ans. (d) When a spring is cut into two, no. of coils gets halved.
$\therefore$ Stiffness of each half gets doubled.
When these are connected in parallel, stiffness = 2k + 2k = 4k
Therefore deflection will be ¼ times. = 2.5 mm

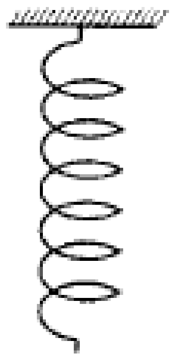
Previous 20-Years IES Questions

Helical spring

IES-1. A helical coil spring with wire diameter 'd' and coil diameter 'D' is subjected to external load. A constant ratio of d and D has to be maintained, such that the extension of spring is independent of d and D. What is this ratio? [IES-2008]

(a) D^3/d^4 (b) d^3/D^4 (c) $\frac{D^{4/3}}{d^3}$ (d) $\frac{d^{4/3}}{D^3}$

IES-1. Ans. (a) $\delta = \frac{8PD^3N}{Gd^4}$



$$T = F \times \frac{D}{2}; \qquad U = \frac{1}{2}T\theta$$

$$T = \frac{FD}{2}; \qquad \theta = \frac{TL}{GJ}$$

$$L = \pi DN$$

$$U = \frac{1}{2}\left(\frac{FD}{2}\right)^2\left(\frac{L}{GJ}\right) = \frac{4F^2D^3N}{Gd^4}$$

$$\delta = \frac{\partial U}{\partial F} = \frac{8FD^3N}{Gd^4}$$

IES-2. Assertion (A): Concentric cylindrical helical springs are used to have greater spring force in a limited space. [IES-2006]
Reason (R): Concentric helical springs are wound in opposite directions to prevent locking of coils under heavy dynamic loading.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-2. Ans. (b)

IES-3. Assertion (A): Two concentric helical springs used to provide greater spring force are wound in opposite directions. [IES-1995; IAS-2004]
Reason (R): The winding in opposite directions in the case of helical springs prevents buckling.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A

(c) A is true but R is false
(d) A is false but R is true

IES-3. Ans. (c) It is for preventing locking not for buckling.

IES-4. Which one of the following statements is correct? [IES-1996; 2007; IAS-1997]
If a helical spring is halved in length, its spring stiffness
(a) Remains same (b) Halves (c) Doubles (d) Triples

IES-4. Ans. (c) Stiffness of $\text{sprin}(k) = \dfrac{Gd^4}{8D^3n}$ so $k \propto \dfrac{1}{n}$ and n wiil be half

IES-5. A body having weight of 1000 N is dropped from a height of 10 cm over a close-coiled helical spring of stiffness 200 N/cm. The resulting deflection of spring is nearly [IES-2001]
(a) 5 cm (b) 16 cm (c) 35 cm (d) 100 cm

IES-5. Ans. (b) $mg(h + x) = \dfrac{1}{2}kx^2$

IES-6. A close-coiled helical spring is made of 5 mm diameter wire coiled to 50 mm mean diameter. Maximum shear stress in the spring under the action of an axial force is 20 N/mm². The maximum shear stress in a spring made of 3 mm diameter wire coiled to 30 mm mean diameter, under the action of the same force will be nearly [IES-2001]
(a) 20 N/mm² (b) 33.3 N/mm² (c) 55.6 N/mm² (d) 92.6 N/mm²

IES-6. Ans. (c) Use $\tau = k_s \dfrac{8PD}{\pi d^3}$

IES-7. A closely-coiled helical spring is acted upon by an axial force. The maximum shear stress developed in the spring is τ. Half of the length of the spring is cut off and the remaining spring is acted upon by the same axial force. The maximum shear stress in the spring the new condition will be: [IES-1995]
(a) ½ τ (b) τ (c) 2 τ (d) 4 τ

IES-7. Ans. (b) Use $\tau = k_s \dfrac{8PD}{\pi d^3}$ it is independent of number of turn

IES-8. The maximum shear stress occurs on the outermost fibers of a circular shaft under torsion. In a close coiled helical spring, the maximum shear stress occurs on the [IES-1999]
(a) Outermost fibres (b) Fibres at mean diameter (c) Innermost fibres (d) End coils

IES-8. Ans. (c)

IES-9. A helical spring has N turns of coil of diameter D, and a second spring, made of same wire diameter and of same material, has N/2 turns of coil of diameter 2D. If the stiffness of the first spring is k, then the stiffness of the second spring will be: [IES-1999]
(a) k/4 (b) k/2 (c) 2k (d) 4k

IES-9. Ans. (a) Stiffness (k) $= \dfrac{Gd^4}{64R^3N}$; Second spring, stiffness $(k_2) = \dfrac{Gd^4}{64(2R)^3 \times \dfrac{N}{2}} = \dfrac{k}{4}$

IES-10. A closed-coil helical spring is subjected to a torque about its axis. The spring wire would experience a [IES-1996; 1998]
(a) Bending stress
(b) Direct tensile stress of uniform intensity at its cross-section
(c) Direct shear stress
(d) Torsional shearing stress

IES-10. Ans. (a)

IES-11. Given that: [IES-1996]
d = diameter of spring, R = mean radius of coils, n = number of coils and G = modulus of rigidity, the stiffness of the close-coiled helical spring subject to an axial load W is equal to

(a) $\dfrac{Gd^4}{64R^3n}$ (b) $\dfrac{Gd^3}{64R^3n}$ (c) $\dfrac{Gd^4}{32R^3n}$ (d) $\dfrac{Gd^4}{64R^2n}$

IES-11. Ans. (a)

IES-12. A closely coiled helical spring of 20 cm mean diameter is having 25 coils of 2 cm diameter rod. The modulus of rigidity of the material if 10^7 N/cm². What is the stiffness for the spring in N/cm? [IES-2004]

(a) 50 (b) 100 (c) 250 (d) 500

IES-12. Ans. (b) Stiffness of sprin$(k) = \dfrac{Gd^4}{8D^3n} = \dfrac{10^7\left(N/cm^2\right)\times 2^4\left(cm^4\right)}{8\times 20^3\left(cm^3\right)\times 25} = 100\,N/cm$

IES-13. Which one of the following expresses the stress factor K used for design of closed coiled helical spring? [IES-2008]

(a) $\dfrac{4C-4}{4C-1}$ (b) $\dfrac{4C-1}{4C-4}+\dfrac{0.615}{C}$ (c) $\dfrac{4C-4}{4C-1}+\dfrac{0.615}{C}$ (d) $\dfrac{4C-1}{4C-4}$

Where C = spring index

IES-13. Ans. (b)

IES-14. In the calculation of induced shear stress in helical springs, the Wahl's correction factor is used to take care of [IES-1995; 1997]

(a) Combined effect of transverse shear stress and bending stresses in the wire.
(b) Combined effect of bending stress and curvature of the wire.
(c) Combined effect of transverse shear stress and curvature of the wire.
(d) Combined effect of torsional shear stress and transverse shear stress in the wire.

IES-14. Ans. (c)

IES-15. While calculating the stress induced in a closed coil helical spring, Wahl's factor must be considered to account for [IES-2002]

(a) The curvature and stress concentration effect (b) Shock loading
(c) Poor service conditions (d) Fatigue loading

IES-15. Ans. (a)

IES-16. Cracks in helical springs used in Railway carriages usually start on the inner side of the coil because of the fact that [IES-1994]

(a) It is subjected to the higher stress than the outer side.
(b) It is subjected to a higher cyclic loading than the outer side.
(c) It is more stretched than the outer side during the manufacturing process.
(d) It has a lower curvature than the outer side.

IES-16. Ans. (a)

IES-17. Two helical springs of the same material and of equal circular cross-section and length and number of turns, but having radii 20 mm and 40 mm, kept concentrically (smaller radius spring within the larger radius spring), are compressed between two parallel planes with a load P. The inner spring will carry a load equal to [IES-1994]

(a) P/2 (b) 2P/3 (c) P/9 (d) 8P/9

IES-17. Ans. (d) $\dfrac{W_o}{W_i} = \dfrac{R_i^3}{R_o^3} = \left(\dfrac{20}{40}\right)^3 = \dfrac{1}{8};\ W_o = \dfrac{W_i}{8}\ \ So\ W_i + \dfrac{W_i}{8} = P\ or\ W_i = \dfrac{8}{9}P$

IES-18. A length of 10 mm diameter steel wire is coiled to a close coiled helical spring having 8 coils of 75 mm mean diameter, and the spring has a stiffness K. If the same length of wire is coiled to 10 coils of 60 mm mean diameter, then the spring stiffness will be: [IES-1993]

(a) K (b) 1.25 K (c) 1.56 K (d) 1.95 K

IES-18. Ans. (c) $\text{Stiffness of spring (k)} = \dfrac{Gd^4}{64R^3n}$ Where G and d is same

$$\text{Therefore } \frac{k}{k_2} = \frac{1}{\left(\frac{R}{R_2}\right)^3\left(\frac{n}{n_2}\right)} = \frac{1}{\left(\frac{75}{60}\right)^3\left(\frac{8}{10}\right)} = \frac{1}{1.56}$$

IES-19. A spring with 25 active coils cannot be accommodated within a given space. Hence 5 coils of the spring are cut. What is the stiffness of the new spring?
(a) Same as the original spring (b) 1.25 times the original spring **[IES-2004]**
(c) 0.8 times the original spring (d) 0.5 times the original spring

IES-19. Ans. (b) Stiffness of spring $(k) = \frac{Gd^4}{8D^3n} \quad \therefore k\alpha\frac{1}{n} \qquad \text{or } \frac{k_2}{k_1} = \frac{n_1}{n_2} = \frac{25}{20} = 1.25$

IES-20. Wire diameter, mean coil diameter and number of turns of a closely-coiled steel spring are d, D and N respectively and stiffness of the spring is K. A second spring is made of same steel but with wire diameter, mean coil diameter and number of turns 2d, 2D and 2N respectively. The stiffness of the new spring is:
[IES-1998; 2001]
(a) K (b) 2K (c) 4K (d) 8K

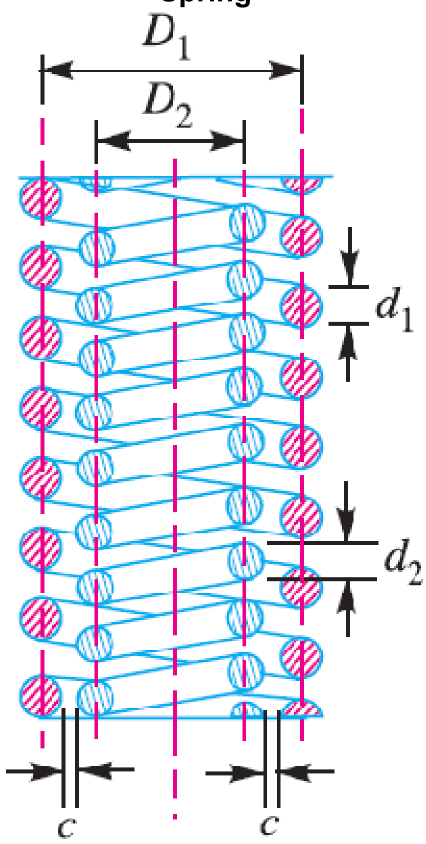
IES-20. Ans. (a) Stiffness of spring $(k) = \frac{Gd^4}{8D^3n}$

IES-21. When two springs of equal lengths are arranged to form cluster springs which of the following statements are the: **[IES-1992]**
1. Angle of twist in both the springs will be equal
2. Deflection of both the springs will be equal
3. Load taken by each spring will be half the total load
4. Shear stress in each spring will be equal
(a) 1 and 2 only (b) 2 and 3 only (c) 3 and 4 only (d) 1, 2 and 4 only

IES-21. Ans. (a)

IES-22. Consider the following statements: **[IES-2009]**
When two springs of equal lengths are arranged to form a cluster spring
1. Angle of twist in both the springs will be equal
2. Deflection of both the springs will be equal
3. Load taken by each spring will be half the total load
4. Shear stress in each spring will be equal
Which of the above statements is/are correct?
(a) 1 and 2 (b) 3 and 4 (c)2 only (d) 4 only

IES-22. Ans. (a) Same as [IES-1992]



Close-coiled helical spring with axial load

IES-23. Under axial load, each section of a close-coiled helical spring is subjected to **[IES-2003]**
(a) Tensile stress and shear stress due to load
(b) Compressive stress and shear stress due to torque
(c) Tensile stress and shear stress due to torque
(d) Torsional and direct shear stresses

IES-23. Ans. (d)

IES-24. When a weight of 100 N falls on a spring of stiffness 1 kN/m from a height of 2 m, the deflection caused in the first fall is: **[IES-2000]**
(a) Equal to 0.1 m (b) Between 0.1 and 0.2 m
(c) Equal to 0.2 m (d) More than 0.2 m

IES-24. Ans. (d) use $mg(h+x)=\frac{1}{2}kx^2$

Subjected to 'Axial twist'

IES-25. A closed coil helical spring of mean coil diameter 'D' and made from a wire of diameter 'd' is subjected to a torque 'T' about the axis of the spring. What is the maximum stress developed in the spring wire? **[IES-2008]**
(a) $\frac{8T}{\pi d^3}$ (b) $\frac{16T}{\pi d^3}$ (c) $\frac{32T}{\pi d^3}$ (d) $\frac{64T}{\pi d^3}$

IES-25. Ans. (b)

Springs in Series

IES-26. When a helical compression spring is cut into two equal halves, the stiffness of each of the result in springs will be: **[IES-2002; IAS-2002]**
(a) Unaltered (b) Double (c) One-half (d) One-fourth

IES-26. Ans. (b)

IES-27. If a compression coil spring is cut into two equal parts and the parts are then used in parallel, the ratio of the spring rate to its initial value will be: **[IES-1999]**

(a) 1 (b) 2 (c) 4 (d) Indeterminable for want of sufficient data

IES-27. Ans. (c) When a spring is cut into two, no. of coils gets halved.

$\therefore$ Stiffness of each half gets doubled.

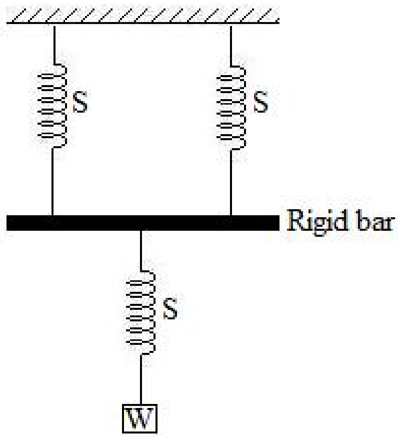
When these are connected in parallel, stiffness = 2k + 2k = 4k

Springs in Parallel

IES-28. The equivalent spring stiffness for the system shown in the given figure (S is the spring stiffness of each of the three springs) is:

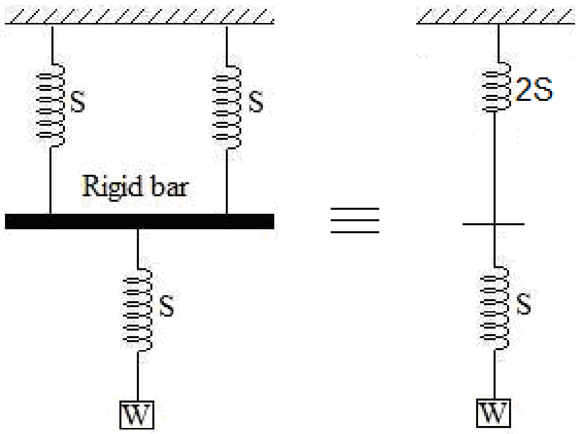
(a) S/2 (b) S/3

(c) 2S/3 (d) S



[IES-1997; IAS-2001]

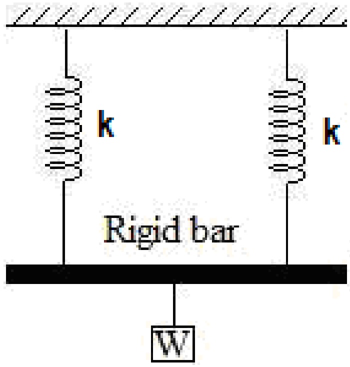
IES-28. Ans. (c) $\frac{1}{S_e} = \frac{1}{2S} + \frac{1}{S} \quad or\ S_e = \frac{2}{3}S$



IES-29. Two coiled springs, each having stiffness K, are placed in parallel. The stiffness of the combination will be: [IES-2000]

(a) $4K$ (b) $2K$ (c) $\frac{K}{2}$ (d) $\frac{K}{4}$

IES-29. Ans. (b) $W = k\delta = k_1\delta + k_2\delta$



IES-30. A mass is suspended at the bottom of two springs in series having stiffness 10 N/mm and 5 N/mm. The equivalent spring stiffness of the two springs is nearly [IES-2000]

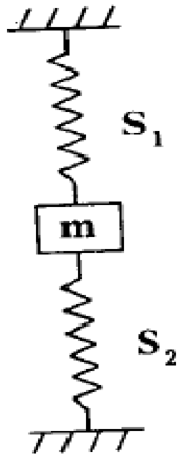
(a) 0.3 N/mm (b) 3.3 N/mm (c) 5 N/mm (d) 15 N/mm

IES-30. Ans. (b) $\frac{1}{S_e} = \frac{1}{10} + \frac{1}{5} \quad or\ S_e = \frac{10}{3}$

IES-31. **Figure given above shows a spring-mass system where the mass m is fixed in between two springs of stiffness S_1 and S_2. What is the equivalent spring stiffness?**

(a) $S_1 - S_2$ (b) $S_1 + S_2$

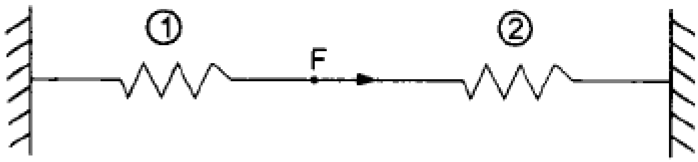
(c) $(S_1 + S_2)/S_1 S_2$ (d) $(S_1 - S_2)/S_1 S_2$



[IES-2005]

IES-31. Ans. (b)

IES-32. **Two identical springs labelled as 1 and 2 are arranged in series and subjected to force F as shown in the given figure.**



Assume that each spring constant is K. The strain energy stored in spring 1 is: **[IES-2001]**

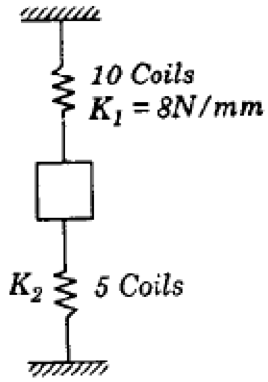
(a) $\dfrac{F^2}{2K}$ (b) $\dfrac{F^2}{4K}$ (c) $\dfrac{F^2}{8K}$ (d) $\dfrac{F^2}{16K}$

IES-32. Ans. (c) The strain energy stored per spring $= \frac{1}{2}k.x^2/2 = \frac{1}{2}\times k_{eq}\times\left(\frac{F}{k_{eq}}\right)^2/2$ and here total force 'F' is supported by both the spring 1 and 2 therefore $k_{eq} = k + k = 2k$

IES-33. **What is the equivalent stiffness (i.e. spring constant) of the system shown in the given figure?**

(a) 24 N/mm (b) 16 N/mm

(c) 4 N/mm (d) 5.3 N/mm



[IES-1997]

IES-33. Ans. (a) Stiffness K_1 of 10 coils spring = 8 N/mm

$\therefore$ Stiffness K_2 of 5 coils spring = 16 N/mm

Though it looks like in series but they are in parallel combination. They are not subjected to same force. Equivalent stiffness (k) = $k_1 + k_2$ = 24 N/mm

Previous 20-Years IAS Questions

Helical spring

IAS-1. **Assertion (A): Concentric cylindrical helical springs which are used to have greater spring force in a limited space is wound in opposite directions.**

Reason (R): Winding in opposite directions prevents locking of the two coils in case of misalignment or buckling. **[IAS-1996]**

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-1. Ans. (a)

IAS-2. **An open-coiled helical spring of mean diameter D, number of coils N and wire diameter d is subjected to an axial force' P. The wire of the spring is subject to:** **[IAS-1995]**

(a) direct shear only (b) combined shear and bending only
(c) combined shear, bending and twisting (d) combined shear and twisting only

IAS-2. Ans. (d)

IAS-3. **Assertion (A): Two concentric helical springs used to provide greater spring force are wound in opposite directions.** **[IES-1995; IAS-2004]**
Reason (R): The winding in opposite directions in the case of helical springs prevents buckling.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-3. Ans. (c) It is for preventing locking not for buckling.

IAS-4. **Which one of the following statements is correct?** **[IES-1996; 2007; IAS-1997]**
If a helical spring is halved in length, its spring stiffness

(a) Remains same (b) Halves (c) Doubles (d) Triples

IAS-4. Ans. (c) Stiffness of $\text{sprin}(k) = \dfrac{Gd^4}{8D^3n}$ so $k \propto \dfrac{1}{n}$ and n wiil be half

IAS-5. **A closed coil helical spring has 15 coils. If five coils of this spring are removed by cutting, the stiffness of the modified spring will:** **[IAS-2004]**

(a) Increase to 2.5 times (b) Increase to 1.5 times
(c) Reduce to 0.66 times (d) Remain unaffected

IAS-5. Ans. (b) $K=\dfrac{Gd^4}{8D^3N}$ $\quad or\ K\,\alpha\,\dfrac{1}{N}\quad or\ \dfrac{K_2}{K_1}=\dfrac{N_1}{N_2}=\dfrac{15}{10}=1.5$

IAS-6. **A close-coiled helical spring has wire diameter 10 mm and spring index 5. If the spring contains 10 turns, then the length of the spring wire would be:** **[IAS-2000]**

(a) 100 mm (b) 157 mm (c) 500 mm (d) 1570 mm

IAS-6. Ans. (d) $l=\pi Dn=\pi\left(cd\right)n=\pi\times\left(5\times10\right)\times10=1570\,mm$

IAS-7. **Consider the following types of stresses:** **[IAS-1996]**
1. torsional shear 2. Transverse direct shear 3. Bending stress
The stresses, that are produced in the wire of a close-coiled helical spring subjected to an axial load, would include

(a) 1 and 3 (b) 1 and 2 (c) 2 and 3 (d) 1, 2 and 3

IAS-7. Ans. (b)

IAS-8. **Two close-coiled springs are subjected to the same axial force. If the second spring has four times the coil diameter, double the wire diameter and double the number of coils of the first spring, then the ratio of deflection of the second spring to that of the first will be:** **[IAS-1998]**

(a) 8 (b) 2 (c) $\dfrac{1}{2}$ (d) 1/16

IAS-8. Ans. (a) $\delta = \dfrac{8PD^3N}{Gd^4}$ or $\dfrac{\delta_2}{\delta_1} = \dfrac{\left(\dfrac{D_2}{D_1}\right)\left(\dfrac{N_2}{N_1}\right)}{\left(\dfrac{d_2}{d_1}\right)^4} = \dfrac{4^3 \times 2}{2^4} = 8$

IAS-9. A block of weight 2 N falls from a height of 1m on the top of a spring· If the spring gets compressed by 0.1 m to bring the weight momentarily to rest, then the spring constant would be: [IAS-2000]

(a) 50 N/m (b) 100 N/m (c) 200N/m (d) 400N/m

IAS-9. Ans. (d) Kinetic energy of block = potential energy of spring

or $W \times h = \dfrac{1}{2}k.x^2$ or $k = \dfrac{2Wh}{x^2} = \dfrac{2 \times 2 \times 1}{0.1^2} N/m = 400\,N/m$

IAS-10. The springs of a chest expander are 60 cm long when unstretched. Their stiffness is 10 N/mm. The work done in stretching them to 100 cm is: [IAS-1996]

(a) 600 Nm (b) 800 Nm (c) 1000 Nm (d) 1600 Nm

IAS-10. Ans. (b) $E = \dfrac{1}{2}kx^2 = \dfrac{1}{2} \times \left\{ \dfrac{10N}{\left(\dfrac{1}{1000}\right)m} \right\} \times \{1-0.6\}^2 m^2 = 800Nm$

IAS-11. A spring of stiffness 'k' is extended from a displacement x_1 to a displacement x_2 the work done by the spring is: [IAS-1999]

(a) $\dfrac{1}{2}k\,x_1^2 - \dfrac{1}{2}k\,x_2^2$ (b) $\dfrac{1}{2}k\,(x_1 - x_2)^2$ (c) $\dfrac{1}{2}k\,(x_1 + x_2)^2$ (d) $k\left(\dfrac{x_1 + x_2}{2}\right)^2$

IAS-11. Ans. (a) Work done by the spring is = $\dfrac{1}{2}k\,x_1^2 - \dfrac{1}{2}k\,x_2^2$

IAS-12. A spring of stiffness 1000 N/m is stretched initially by 10 cm from the undeformed position. The work required to stretch it by another 10 cm is: [IAS-1995]

(a) 5 Nm (b) 7 Nm (c) 10 Nm (d) 15 Nm.

IAS-12. Ans. (d) $E = \dfrac{1}{2}k\left(x_2^2 - x_1^2\right) = \dfrac{1}{2} \times 1000 \times \{0.20^2 - 0.10^2\} = 15Nm$

Springs in Series

IAS-13. When a helical compression spring is cut into two equal halves, the stiffness of each of the result in springs will be: [IES-2002; IAS-2002]

(a) Unaltered (b) Double (c) One-half (d) One-fourth

IAS-13. Ans. (b)

IAS-14. The length of the chest-expander spring when it is un-stretched, is 0.6 m and its stiffness is 10 N/mm. The work done in stretching it to 1m will be: [IAS-2001]

(a) 800 J (b) 1600 J (c) 3200 J (d) 6400 J

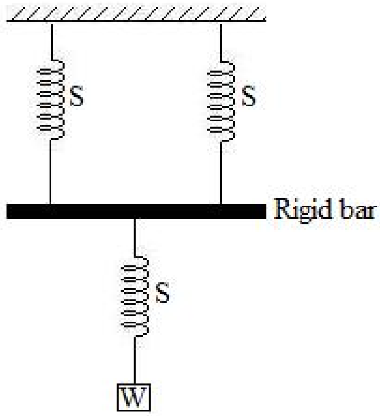
IAS-14. Ans. (a)

Work done $= \dfrac{1}{2}k.x^2 = \dfrac{1}{2} \times \left(\dfrac{10N}{1mm}\right) \times (1-0.6)^2 m^2 = \dfrac{1}{2} \times \dfrac{10N}{\left(\dfrac{1}{1000}\right)m} \times 0.4^2 m^2 = 800J$

Springs in Parallel

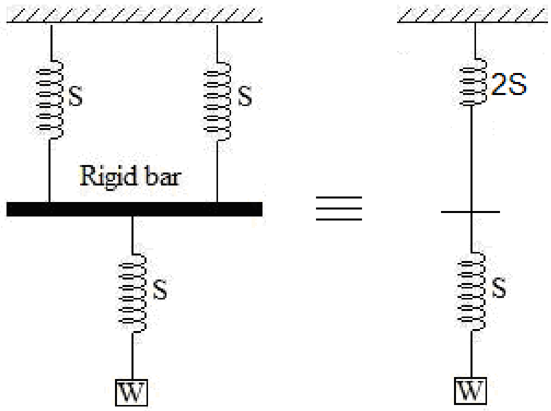
IAS-15. The equivalent spring stiffness for the system shown in the given figure (S is the spring stiffness of each of the three springs) is:

(a) S/2 (b) S/3
(c) 2S/3 (d) S



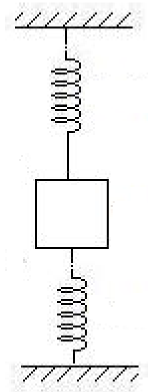
[IES-1997; IAS-2001]

IAS-15. Ans. (c) $\frac{1}{S_e} = \frac{1}{2S} + \frac{1}{S}$ or $S_e = \frac{2}{3}S$



IAS-16. Two identical springs, each of stiffness K, are assembled as shown in the given figure. The combined stiffness of the assembly is:

(a) K^2 (b) 2K
(c) K (d) (1/2)K



[IAS-1998]

IAS-16. Ans. (b) Effective stiffness = 2K. Due to applied force one spring will be under tension and another one under compression so total resistance force will double.

Flat spiral Spring

IAS-17. Mach List-I (Type of spring) with List-II (Application) and select the correct answer: **[IAS-2000]**

	List-I		**List-II**
A.	Leaf/Helical springs	**1.**	Automobiles/Railways coachers
B.	Spiral springs	**2.**	Shearing machines
C.	Belleville springs	**3.**	Watches

Codes:	**A**	**B**	**C**		**A**	**B**	**C**
(a)	1	2	3	(b)	1	3	2
(c)	3	1	2	(d)	2	3	1

IAS-17. Ans. (b)

Semi-elliptical spring

IAS-18. The ends of the leaves of a semi-elliptical leaf spring are made triangular in plain in order to: [IAS 1994]

(a) Obtain variable I in each leaf
(b) Permit each leaf to act as a overhanging beam
(c) Have variable bending moment in each leaf
(d) Make MiI constant throughout the length of the leaf.

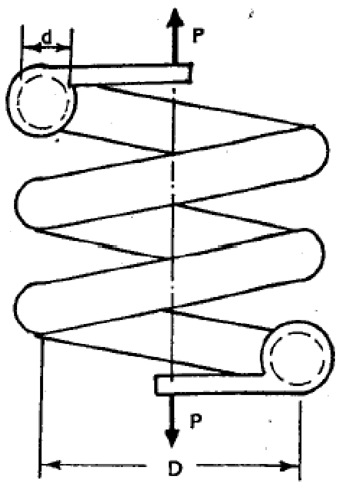
IAS-18. Ans. (d) The ends of the leaves of a semi-elliptical leaf spring are made rectangular in plan in order to make M/I constant throughout the length of the leaf.

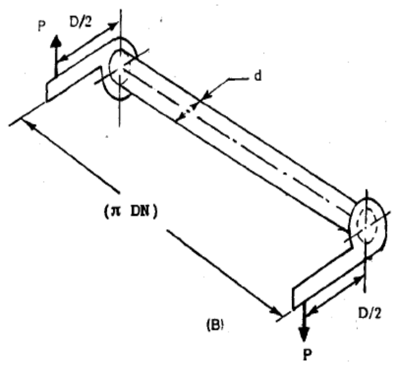
Previous Conventional Questions with Answers

Conventional Question ESE-2008

***Question*:** **A close-coiled helical spring has coil diameter D, wire diameter d and number of turn n. The spring material has a shearing modulus G. Derive an expression for the stiffness k of the spring.**

***Answer*:** The work done by the axial force 'P' is converted into strain energy and stored in the spring.





$$U = (\text{average torque}) \times (\text{angular displacement})$$

$$= \frac{T}{2} \times \theta$$

From the figure we get, $\theta = \frac{TL}{GJ}$

Torque $(T) = \frac{PD}{2}$

length of wire $(L) = \pi D n$

Polar moment of Inertia$(J) = \frac{\pi d^4}{32}$

Therefore $U = \frac{4P^2D^3n}{Gd^4}$

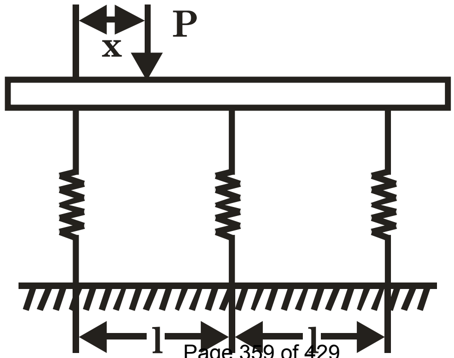
According to Castigliano's theorem, the displacement corresponding to force P is obtained by partially differentiating strain energy with respect to that force.

$$\text{Therefore } \delta = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left[\frac{4p^2D^3n}{Gd^4}\right] = \frac{8PD^3n}{Gd^4}$$

$$\text{So Spring stiffness, } (k) = \frac{P}{\delta} = \frac{Gd^4}{8D^3n}$$

Conventional Question ESE-2010

Q. **A stiff bar of negligible weight transfers a load P to a combination of three helical springs arranged in parallel as shown in the above figure. The springs are made up of the same material and out of rods of equal diameters. They are of same free length before loading. The number of coils in those three springs are 10, 12 and 15 respectively, while the mean coil diameters are in ratio of 1 : 1.2 : 1.4 respectively. Find the distance 'x' as shown in figure, such that the stiff bar remains horizontal after the application of load P.** **[10 Marks]**



Ans. Same free length of spring before loading

The number of coils in the spring 1,2 and 3 is 10, 12 and 15 mean diameter of spring 1,2 and 3 in the ratio of 1 : 1.2 : 1.4 Find out distance x so that rod remains horizontal after loading.

Since the rod is rigid and remains horizontal after the load p is applied therefore the deflection of each spring will be same

$$\delta_1 = \delta_2 = \delta_3 = \delta \qquad \text{(say)}$$

Spring are made of same material and out of the rods of equal diameter

$$G_1 = G_2 = G_3 = G \ \text{ and } \ d_1 = d_2 = d_3 = d$$

Load in spring 1

$$P_1 = \frac{Gd^4\delta}{64R_1^3 n_1} = \frac{Gd^4\delta}{64R_1^3 \times 10} = \frac{Gd^4\delta}{640R_1^3} \quad .....(1)$$

Load in spring 2

$$P_2 = \frac{Gd^4\delta}{64 \times R_2^3 n_2} = \frac{Gd^4\delta}{64 \times (1.2)^3 \times 12R_1^3} = \frac{Gd^4\delta}{1327.10R_1^3} \quad .....(2)$$

Load in spring 3

$$P_3 = \frac{Gd^4\delta}{64R_3^3 n_3} = \frac{Gd^4\delta}{64 \times (1.4)^3 \times 15R_1^3} = \frac{Gd^4\delta}{2634.2R_1^3} \quad .....(3)$$

From eq^n (1) & (2)

$$P_2 = \frac{640}{1327.1} P_1$$

$$P_2 = 0.482\, P_1$$

from eq^n (1) & (3)

$$P_3 = \frac{640}{2634.2} \ P_1 = 0.2430\, P_1$$

Taking moment about the line of action P_1

$$P_2 \times L + P_3 \times 2L = P.x$$

$$0.4823 \ P_1 L + 0.2430 \ P_1 \times 2L = P.x.$$

$$x = \frac{(0.4823 + 0.486)P_1 L}{P} \qquad ....(4)$$

total load in the rod is

$$P = P_1 + P_2 + P_3$$

$$P = P_1 + .4823P_1 + 0.2430P_1$$

$$P = 1.725 \ P_1 \qquad ......(5)$$

Equation (4) & (5)

$$x = \frac{0.9683L}{1.725 \ P_1 / P_1} = \frac{0.9683L}{1.725} = 0.5613L$$

$$x = 0.5613 \ L$$

Conventional Question ESE-2008

Question: **A close-coiled helical spring has coil diameter to wire diameter ratio of 6. The spring deflects 3 cm under an axial load of 500N and the maximum shear stress is not to exceed 300 MPa. Find the diameter and the length of the spring wire required. Shearing modulus of wire material = 80 GPa.**

Answer: $Stiffness, K = \frac{P}{\delta} = \frac{Gd^4}{8D^3 n}$

$$or, \frac{500}{0.03} = \frac{\left(80\times10^{9}\right)\times d}{8\times6^{3}\times n} \qquad [given\ c=\frac{D}{d}=6]$$

$$or, d = 3.6\times10^{-4}n ---(i)$$

For static loading correcting factor(k)

$$k=\left(1+\frac{0.5}{c}\right)=\left(1+\frac{0.5}{6}\right)=1.0833$$

We know that $(\tau)=k\frac{8PD}{\pi d^3}$

$$d^2 = \frac{8kPC}{\pi\tau} \qquad \left[\because C=\frac{D}{d}=6\right]$$

$$d=\sqrt{\frac{1.0833\times8\times500\times6}{\pi\times300\times10^{6}}}=5.252\times10^{-3}m=5.252mm$$

So D=cd=6×5.252mm=31.513mm

From, equation (i) $\qquad$ n=14.59 $\simeq$ 15

Now length of spring wire(L) =πDn =$\pi\times$31.513×15 mm =1.485 m

Conventional Question ESE-2007

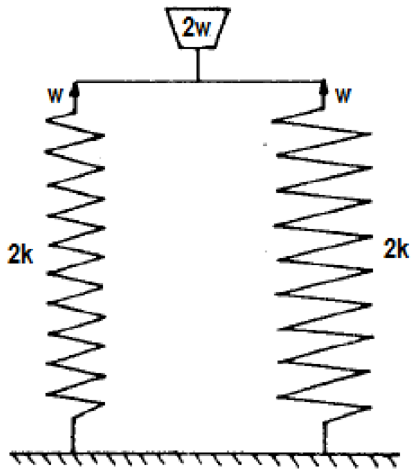
Question: **A coil spring of stiffness 'k' is cut to two halves and these two springs are assembled in parallel to support a heavy machine. What is the combined stiffness provided by these two springs in the modified arrangement?**

Answer: When it cut to two halves stiffness of each half will be 2k. Springs in parallel. Total load will be shared so

Total load = W+W

$$or\ \delta.K_{eq}=\delta.(2k)+\delta.(2k)$$

$$or\ K_{eq}=4k.$$



Conventional Question ESE-2001

Question: **A helical spring B is placed inside the coils of a second helical spring A , having the same number of coils and free axial length and of same material. The two springs are compressed by an axial load of 210 N which is shared between them. The mean coil diameters of A and B are 90 mm and 60 mm and the wire diameters are 12 mm and 7 mm respectively. Calculate the load shared by individual springs and the maximum stress in each spring.**

Answer: The stiffness of the spring (k) =$\frac{Gd^4}{8D^3N}$

Here load shared the springs are arranged in parallel

Equivalent stiffness$(k_e)=k_A+k_B$

$$Hear\ \frac{K_A}{K_B}=\left(\frac{d_A}{d_B}\right)^4\left(\frac{D_B}{D_A}\right)^3\ [As\ N_A=N_N]=\left(\frac{12}{7}\right)^4\times\left(\frac{60}{90}\right)^3=2.559$$

Let total deflection is 'x' m $\qquad x=\frac{\text{Total load}}{\text{Equivalet stiffness}}=\frac{210N}{K_A+K_B}$

$$\text{Load shared by spring 'A'}(F_A) = K_A \times x = \frac{210}{\left(1+\dfrac{k_B}{k_A}\right)} = \frac{210}{\left(1+\dfrac{1}{2.559}\right)} = 151\,\text{N}$$

$$\text{Load shared by spring 'A'}(F_B) = K_B \times x = (210-151) = 59\,\text{N}$$

$$\text{For static load: } \tau = \left(1+\frac{0.5}{C}\right)\frac{8PD}{\pi d^3}$$

$$(\tau_A)_{\max} = \left\{1+\frac{0.5}{\left(\dfrac{90}{12}\right)}\right\}\frac{8\times 151\times 0.090}{\pi\times(0.012)^3} = 21.362\,\text{MPa}$$

$$(\tau_B)_{\max} = \left(1+\frac{0.5}{\left(\dfrac{60}{7}\right)}\right)\frac{8\times 59\times 0.060}{\pi\times(0.007)^3} = 27.816\ \text{MPa}$$

Conventional Question AMIE-1997

***Question*:** **A close-coiled spring has mean diameter of 75 mm and spring constant of 90 kN/m. It has 8 coils. What is the suitable diameter of the spring wire if maximum shear stress is not to exceed 250 MN/m²? Modulus of rigidity of the spring wire material is 80 GN/m². What is the maximum axial load the spring can carry?**

***Answer*:** Given $D = 75\,\text{mm};\ k = 80\,\text{kN/m};\ n = 8$

$\tau = 250\,\text{MN/m}^2;\ G = 80\,\text{GN/m}^2 = 80\times 10^9\,\text{N/m}^2$

Diameter of the spring wire, d:

$$T = \tau \times \frac{\pi}{16} d^3 \qquad (\text{where } T = P\times R)$$

We know, $$P\times 0.0375 = \left(250\times 10^6\right)\times\frac{\pi}{16}d^3 \qquad ---(i)$$

Also $P = k\delta$

or $$P = 80\times 10^3\times\delta \qquad ---(ii)$$

Using the relation:

$$\delta = \frac{8PD^3n}{Gd^4} = \frac{8P\times(0.075)^3\times 8}{80\times 10^9\times d^4} = 33.75\times 10^{-14}\times\frac{P}{d^4}$$

Substituting for δ in equation (ii), we get

$$P = 80\times 10^3\times 33.75\times 10^{-14}\times\frac{P}{d^4} \quad \text{or} \quad d = 0.0128\,\text{m or } 12.8\,\text{mm}$$

Maximum axial load the spring can carry P:
From equation (i), we get

$$P\times 0.0375 = \left(250\times 10^6\right)\times\frac{\pi}{16}\times(0.0128)^3; \qquad \therefore\ P = 2745.2\,\text{N} = 2.7452\,\text{kN}$$

13. Theories of Column

Theory at a Glance (for IES, GATE, PSU)

1. Introduction

- **Strut:** A member of structure which carries an axial compressive load.
- **Column:** If the strut is vertical it is known as column.
- A long, slender column becomes unstable when its axial compressive load reaches a value called the critical buckling load.
- If a beam element is under a compressive load and its length is an order of magnitude larger than either of its other dimensions such a beam is called a *columns*.
- Due to its size its axial displacement is going to be very small compared to its lateral deflection called *buckling*.
- *Buckling* does not vary linearly with load it occurs suddenly and is therefore dangerous
- **Slenderness Ratio:** The ratio between the length and least radius of gyration.
- **Elastic Buckling:** Buckling with no permanent deformation.
- Euler buckling is only valid for long, slender objects in the elastic region.
- For short columns, a different set of equations must be used.

2. Which is the critical load?

- At this value the structure is in equilibrium regardless of the magnitude of the angle (provided it stays small)
- Critical load is the only load for which the structure will be in equilibrium in the disturbed position
- At this value, restoring effect of the moment in the spring matches the buckling effect of the axial load represents the boundary between the stable and unstable conditions.
- If the axial load is less than P_{cr} the effect of the moment in the spring dominates and the structure returns to the vertical position after a small disturbance – stable condition.
- If the axial load is larger than P_{cr} the effect of the axial force predominates and the structure buckles – unstable condition.
- Because of the large deflection caused by buckling, the least moment of inertia *I* can be expressed as, $I = Ak^2$
- Where: *A* is the cross sectional area and *r* is the *radius of gyration* of the cross sectional area, i.e. $k_{min} = \sqrt{\dfrac{I_{min}}{A}}$

- Note that the *smallest* radius of gyration of the column, i.e. the *least* moment of inertia I should be taken in order to find the critical stress. l/k is called the *slenderness ratio*, it is a measure of the column's flexibility.

3. Euler's Critical Load for Long Column

Assumptions:

(i) The column is perfectly straight and of uniform cross-section

(ii) The material is homogenous and isotropic

(iii) The material behaves elastically

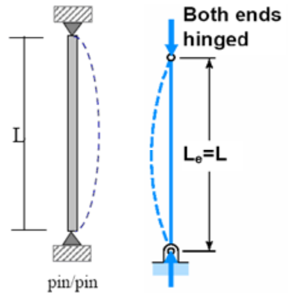
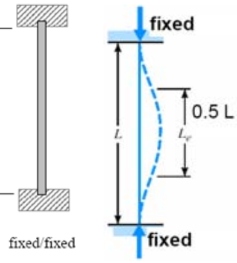
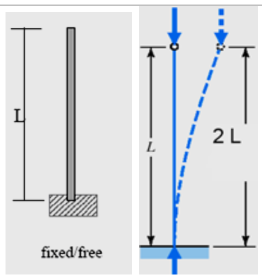
(iv) The load is perfectly axial and passes through the centroid of the column section.

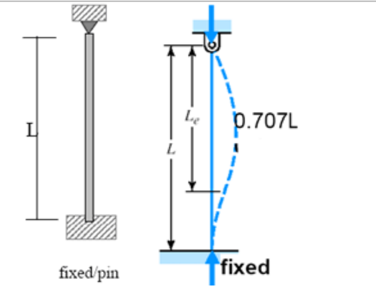
(v) The weight of the column is neglected.

Euler's critical load, $$P_{cr} = \frac{\pi^2 EI}{l_e^2}$$

Where ℓ_e=Equivalent length of column (1st mode of bending)

4. Remember the following table

Case	Diagram	P_{cr}	Equivalent length(l_e)
Both ends hinged/pinned		$\frac{\pi^2 EI}{\ell^2}$	ℓ
Both ends fixed		$\frac{4\pi^2 EI}{\ell^2}$	$\frac{\ell}{2}$
One end fixed & other end free		$\frac{\pi^2 \text{EI}}{4\ell^2}$	2ℓ

One end fixed & other end pinned /hinged		$\dfrac{2\pi^2 EI}{\ell^2}$	$\dfrac{\ell}{\sqrt{2}}$

5. Slenderness Ratio of Column

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \quad \text{where } I = A\ k_{min}^2$$

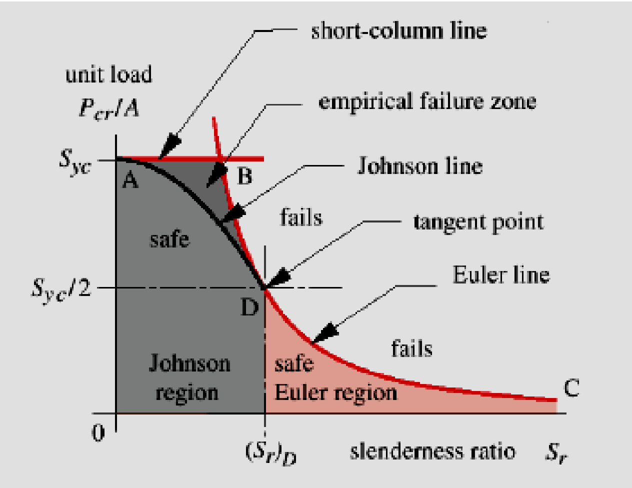
$$= \frac{\pi^2 EA}{\left(\dfrac{\ell_e}{k_{min}}\right)^2} \qquad k_{min} = \text{least radius of gyration}$$

$$\therefore \text{Slenderness Ratio} = \frac{\ell_e}{k_{min}}$$

6. Rankine's Crippling Load

Rankine theory is applied to both

- Short strut /column (valid upto SR-40)
- Long Column (Valid upto SR 120)



Construction of column failure lines

- Slenderness ratio

$$\frac{\ell_e}{k} = \sqrt{\frac{\pi^2 E}{\sigma_e}} \qquad (\sigma_e = \text{critical stress}) = \frac{P_{cr}}{A}$$

- Crippling Load , P

- $$P = \frac{\sigma_c A}{1 + K'\left(\dfrac{\ell_e}{k}\right)^2}$$

where $k' = \text{Rankine constant} = \dfrac{\sigma_c}{\pi^2 E}$ depends on material & end conditions

$\sigma_c = crushing$ stress

- For steel columns

$$K' = \frac{1}{25000} \text{ for both ends fixed}$$

$$= \frac{1}{12500} \text{ for one end fixed \& other hinged} \qquad 20 \le \frac{\ell_e}{k} \le 100$$

7. Other formulas for crippling load (P)

- Gordon's formula,

$$P = \frac{A\sigma_c}{1 + b\left(\frac{\ell_e}{d}\right)^2}$$ b = a constant, d = least diameter or breadth of bar

- Johnson Straight line formula,

$$P = \sigma_c A\left[1 - c\left(\frac{\ell_e}{k}\right)\right]$$ c = a constant depending on material.

- Johnson parabolic formulae :

$$P = \sigma_y A\left[1 - b\left(\frac{l}{k}\right)^2\right]$$

where the value of index 'b' depends on the end conditions.

- Fiddler's formula,

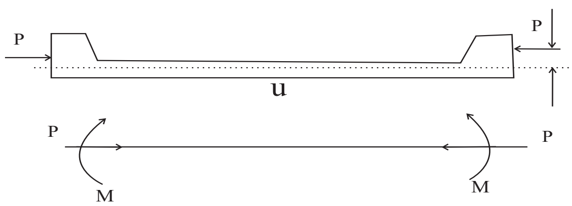
$$P = \frac{A}{C}\left[(\sigma_c + \sigma_e) - \sqrt{\left(\sqrt{\sigma_c + \sigma_e}\right)^2 - 2c\sigma_c\sigma_e}\right]$$

$$\text{where, } \sigma_e = \frac{\pi^2 E}{\left(\ell_e/k\right)^2}$$

8. Eccentrically Loaded Columns

- **Secant formula**

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ey_c}{k^2}\sec\left(\frac{\ell_e}{2k}\right)\sqrt{\frac{P}{EA}}\right]$$



Where σ_{max} =maximum compressive stress

P = load

A = Area of c/s

y_c = Distance of the outermost fiber in compression from the NA

e = Eccentricity of the load

l_e = Equivalent length

k = Radius of gyration $=\sqrt{\dfrac{I}{A}}$

E = Modulus of elasticity of the material

$$M = P.e.Sec\left(\frac{\ell_e}{2k}\sqrt{\frac{P}{EA}}\right)$$

Where M = Moment introduced.

- **Prof. Perry's Formula**

$$\left(\frac{\sigma_{max}}{\sigma_d}-1\right)\left(1-\frac{\sigma_d}{\sigma_e}\right)=\frac{e_1 y_c}{k^2}$$

Where σ_{max} = maximum compressive stress

$$\sigma_d = \frac{P}{A} = \frac{\text{Load}}{\text{c/s area}}$$

$$\sigma_e = \frac{P_e}{A} = \frac{\text{Euler's load}}{c/s \text{ area}}$$

$$p_e = Euler's\ load = \frac{\pi^2 EI}{\ell_e^{\,2}}$$

e' = Versine at mid-length of column due to initial curvature

e = Eccentricity of the load

$e_1 = e' + 1.2e$

y_c = distance of outer most fiber in compression form the NA

k = Radius of gyration

If σ_{max} is allowed to go up to σ_f (permssible stress)

Then, $\eta = \dfrac{e_1 y_c}{k^2}$

$$\sigma_d = \frac{\sigma_f + \sigma_e(1+\eta)}{2} - \sqrt{\left\{\frac{\sigma_f + \sigma_e(1+\eta)}{2}\right\}^2 - \sigma_e\sigma_f}$$

- **Perry-Robertson Formula**

$$\eta = 0.003\left(\frac{\ell_e}{k}\right)$$

$$\sigma_d = \frac{\sigma_f + \sigma_e\left(1+0.003\dfrac{\ell_e}{k}\right)}{2} - \sqrt{\left\{\frac{\sigma_f + \sigma_e(1+0.003\dfrac{\ell_e}{k}}{2}\right\} - \sigma_e\sigma_f}$$

9. ISI's Formula for Columns and Struts

- **For $\frac{\ell_e}{k}$ =0 to 160**

$$P_c = \frac{\sigma_y / fos}{1 + 0.2\sec\left(\frac{\ell_e}{k}\sqrt{\frac{fos \times p_{c'}}{4E}}\right)}$$

Where, P_c = Permissible axial compressive stress

P_c' = A value obtained from above Secant formula

σ_y = Guaranteed minimum yield stress = 2600 kg/cm^2 for mild steel

fos = factor of safety = 1.68

$\frac{l_e}{k}$ = Slenderness ratio

E = Modulus of elasticity = $2.045 \times 10^6\ kg/cm^2$ for mild steel

- **For** $\frac{l_e}{k} > 160$

OBJECTIVE QUESTIONS (GATE, IES, IAS)

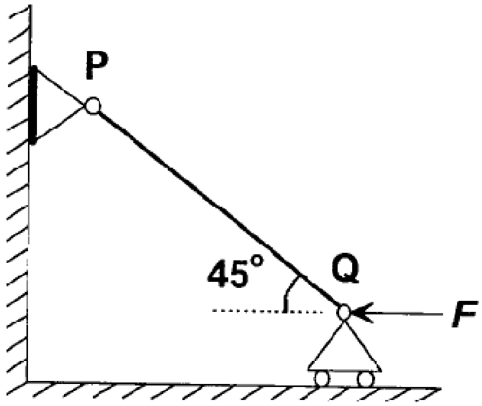
Previous 20-Years GATE Questions

Strength of Column

GATE-1. The rod PQ of length L and with flexural rigidity EI is hinged at both ends. For what minimum force F is it expected to buckle?

(a) $\dfrac{\pi^2 EI}{L^2}$ (b) $\dfrac{\sqrt{2}\pi^2 EI}{L^2}$

(c) $\dfrac{\pi^2 EI}{\sqrt{2L^2}}$ (b) $\dfrac{\pi^2 EI}{2L^2}$



[GATE-2008]

GATE-1. Ans. (b) Axial component of the force F_{PQ} = F Sin 45^0

We know for both end fixed column buckling load (P) $= \dfrac{\pi^2 EI}{L^2}$

and $F\sin 45^0 = P$ or $F = \dfrac{\sqrt{2}\pi^2 EI}{L^2}$

Equivalent Length

GATE-2. The ratio of Euler's buckling loads of columns with the same parameters having (i) both ends fixed, and (ii) both ends hinged is: **[GATE-1998; 2002; IES-2001]**

(a) 2 (b) 4 (c) 6 (d) 8

GATE-2. Ans. (b) Euler's buckling loads of columns

(1) both ends fixed $= \dfrac{4\pi^2 EI}{l^2}$

(2) both ends hinged $= \dfrac{\pi^2 EI}{l^2}$

Euler's Theory (For long column)

GATE-3. A pin-ended column of length L, modulus of elasticity E and second moment of the cross-sectional area I is loaded centrically by a compressive load P. The critical buckling load (P_{cr}) is given by: **[GATE-2006]**

(a) $P_{cr} = \dfrac{EI}{\pi^2 L^2}$ (b) $P_{cr} = \dfrac{\pi^2 EI}{3L^2}$ (c) $P_{cr} = \dfrac{\pi EI}{L^2}$ (d) $P_{cr} = \dfrac{\pi^2 EI}{L^2}$

GATE-3. Ans. (d)

GATE-4. What is the expression for the crippling load for a column of length 'l' with one end fixed and other end free? **[IES-2006; GATE-1994]**

(a) $P = \dfrac{2\pi^2 EI}{l^2}$ (b) $P = \dfrac{\pi^2 EI}{4l^2}$ (c) $P = \dfrac{4\pi^2 EI}{l^2}$ (d) $P = \dfrac{\pi^2 EI}{l^2}$

GATE-4. Ans. (b)

21. The piston rod of diameter 20 mm and length 700 mm in a hydraulic cylinder is subjected to a compressive force of 10 KN due to the internal pressure. The end conditions for the rod may be assumed as guided at the piston end and hinged at the other end. The Young's modulus is 200 GPa. The factor of safety for the piston rod is

(a) 0.68 (b) 2.75 (c) 5.62 (d) 11.0 **[GATE-2007]**

21. Ans. (c)

Assuming guided end to be fixed and other end given as hinged.

The crippling load according to euler's equation,

$$P_{cr} = \frac{2\pi^2 EI}{L^2}, \quad I = \frac{\pi}{64}(20)^4$$

$$= \frac{2\pi^2 \times 200 \times 10^3 \times 7853.9}{(700)^2}$$

$$= 63.27 \text{ kN}$$

$$\text{Factor of safety} = \frac{\text{Crippling load}}{\text{Working load}} = \frac{63.27}{10} = 6.32$$

Previous 20-Years IES Questions

Classification of Column

IES-1. A structural member subjected to an axial compressive force is called **[IES-2008]**

(a) Beam (b) Column (c) Frame (d) Strut

IES-1. Ans. (d) A machine part subjected to an axial compressive force is called a ***strut.*** A strut may be horizontal, inclined or even vertical. But a vertical strut is known as a ***column, pillar*** or ***stanchion.***

The term *column* is applied to all such members except those in which failure would be by simple or pure compression. Columns can be categorized then as:

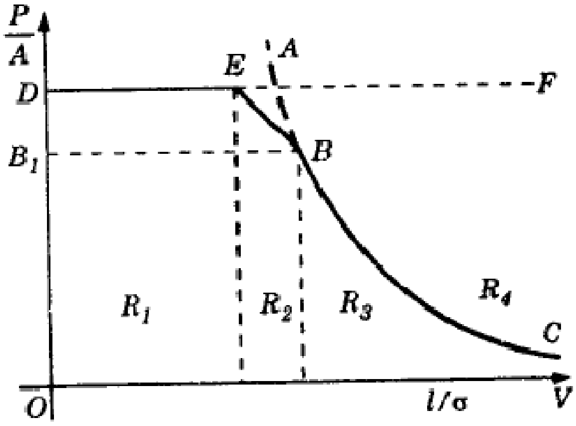
1. Long column with central loading
2. Intermediate-length columns with central loading
3. Columns with eccentric loading
4. Struts or short columns with eccentric loading

IES-2. Which one of the following loadings is considered for design of axles? **[IES-1995]**

(a) Bending moment only
(b) Twisting moment only
(c) Combined bending moment and torsion
(d) Combined action of bending moment, twisting moment and axial thrust.

IES-2. Ans. (a) Axle is a non-rotating member used for supporting rotating wheels etc. and do not transmit any torque. Axle must resist forces applied laterally or transversely to their axes. Such members are called beams.

IES-3. **The curve *ABC* is the Euler's curve for stability of column. The horizontal line *DEF* is the strength limit. With reference to this figure Match List-I with List-II and select the correct answer using the codes given below the lists:**



List-I (Regions)	List-II (Column specification)
A. R_1	**1.** Long, stable
B. R_2	**2.** Short
C. R_3	**3.** Medium
D. R_4	**4.** Long, unstable

[IES-1997]

Codes:	A	B	C	D		A	B	C	D
(a)	2	4	3	1	(b)	2	3	1	4
(c)	1	2	4	3	(d)	2	1	3	4

IES-3. Ans. (b)

IES-4. **Mach List-I with List-II and select the correct answer using the codes given below the lists:** **[IAS-1999]**

List-I	List-II
A. Polar moment of inertia of section	**1.** Thin cylindrical shell
B. Buckling	**2.** Torsion of shafts
C. Neutral axis	**3.** Columns
D. Hoop stress	**4.** Bending of beams

Codes:	A	B	C	D		A	B	C	D
(a)	3	2	1	4	(b)	2	3	4	1
(c)	3	2	4	1	(d)	2	3	1	4

IES-4. Ans. (b)

Strength of Column

IES-5. **Slenderness ratio of a column is defined as the ratio of its length to its**
(a) Least radius of gyration (b) Least lateral dimension **[IES-2003]**
(c) Maximum lateral dimension (d) Maximum radius of gyration

IES-5. Ans. (a)

IES-6. **Assertion (A): A long column of square cross section has greater buckling stability than a similar column of circular cross-section of same length, same material and same area of cross-section with same end conditions.**
Reason (R): A circular cross-section has a smaller second moment of area than a square cross-section of same area. **[IES-1999; IES-1996]**
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-6. Ans. (a)

Equivalent Length

IES-7. **Four columns of same material and same length are of rectangular cross-section of same breadth b. The depth of the cross-section and the end conditions are, however different are given as follows:** **[IES-2004]**

Column	Depth	End conditions
1	0.6 b	Fixed-Fixed
2	0.8 b	Fixed-hinged
3	1.0 b	Hinged-Hinged
4	2.6 b	Fixed-Free

Which of the above columns Euler buckling load maximum?
(a) Column 1 (b) Column 2 (c) Column 3 (d) Column 4

IES-7. Ans. (b)

IES-8. Match List-I (End conditions of columns) with List-II (Equivalent length in terms of length of hinged-hinged column) and select the correct answer using the codes given below the Lists: [IES-2000]

List-I	**List-II**
A. Both ends hinged	**1.** L
B. One end fixed and other end free	**2.** L/ $\sqrt{2}$
C. One end fixed and the other pin-pointed	**3.** 2L
D. Both ends fixed	**4.** L/2

Code:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	1	3	4	2	(b)	1	3	2	4
(c)	3	1	2	4	(d)	3	1	4	2

IES-8. Ans. (b)

IES-9. The ratio of Euler's buckling loads of columns with the same parameters having (i) both ends fixed, and (ii) both ends hinged is: [GATE-1998; 2002; IES-2001]

(a) 2 (b) 4 (c) 6 (d) 8

IES-9. Ans. (b) Euler's buckling loads of columns

$$(1) \text{ both ends fixed} = \frac{4\pi^2 EI}{l^2}$$

$$(2) \text{ both ends hinged} = \frac{\pi^2 EI}{l^2}$$

Euler's Theory (For long column)

IES-10. What is the expression for the crippling load for a column of length 'l' with one end fixed and other end free? [IES-2006; GATE-1994]

(a) $P = \frac{2\pi^2 EI}{l^2}$ (b) $P = \frac{\pi^2 EI}{4l^2}$ (c) $P = \frac{4\pi^2 EI}{l^2}$ (d) $P = \frac{\pi^2 EI}{l^2}$

IES-10. Ans. (b)

IES-11. Euler's formula gives 5 to 10% error in crippling load as compared to experimental results in practice because: [IES-1998]

(a) Effect of direct stress is neglected
(b) Pin joints are not free from friction
(c) The assumptions made in using the formula are not met in practice
(d) The material does not behave in an ideal elastic way in tension and compression

IES-11. Ans. (c)

IES-12. Euler's formula can be used for obtaining crippling load for a M.S. column with hinged ends.

Which one of the following conditions for the slenderness ratio $\frac{l}{k}$ is to be satisfied? [IES-2000]

(a) $5 < \frac{l}{k} < 8$ (b) $9 < \frac{l}{k} < 18$ (c) $19 < \frac{l}{k} < 40$ (d) $\frac{l}{k} \geq 80$

IES-12. Ans. (d)

IES-13. If one end of a hinged column is made fixed and the other free, how much is the critical load compared to the original value? [IES-2008]

(a) ¼ (b) ½ (c) Twice (d) Four times

IES-13. Ans. (a) Critical Load for both ends hinged = $\pi^2 EI/l^2$

And Critical Load for one end fixed, and other end free = $\pi^2 EI/4l^2$

IES-14. If one end of a hinged column is made fixed and the other free, how much is the critical load compared to the original value? [IES-2008]

(a) ¼ (b) ½ (c) Twice (d) Four times

IES-14. Ans. (a) Original load = $\frac{\pi^2 EI}{l^2}$

When one end of hinged column is fixed and other free. New Le = 2L

$\therefore$ New load $= \dfrac{\pi^2 EI}{(2L)^2} = \dfrac{\pi^2 EI}{4L^2} = \dfrac{1}{4} \times \text{Original value}$

IES-15. Match List-I with List-II and select the correct answer using the code given below the Lists: [IES-1995; 2007; IAS-1997]

List-I (Long Column)	**List-II (Critical Load)**
A. Both ends hinged	**1.** $\pi^2 EI/4l^2$
B. One end fixed, and other end free	**2.** $4\pi^2 EI/l^2$
C. Both ends fixed	**3.** $2\pi^2 EI/l^2$
D. One end fixed, and other end hinged	**4.** $\pi^2 EI/l^2$

Code:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	2	1	4	3	(b)	4	1	2	3
(c)	2	3	4	1	(d)	4	3	2	1

IES-15. Ans. (b)

IES-16. The ratio of the compressive critical load for a long column fixed at both the ends and a column with one end fixed and the other end free is: [IES-1997]

(a) 1 : 2 (b) 1: 4 (c) 1: 8 (d) 1: 16

IES-16. Ans. (d) Critical Load for one end fixed, and other end free is $\pi^2 EI/4l^2$ and both ends fixed is $4\pi^2 EI/l^2$

IES-17. The buckling load will be maximum for a column, if [IES-1993]

(a) One end of the column is clamped and the other end is free
(b) Both ends of the column are clamped
(c) Both ends of the column are hinged
(d) One end of the column is hinged and the other end is free

IES-17. Ans. (b) Buckling load of a column will be maximum when both ends are fixed

IES-18. If diameter of a long column is reduced by 20%, the percentage of reduction in Euler buckling load is: [IES-2001]

(a) 4 (b) 36 (c) 49 (d) 59

IES-18. Ans. (d) $P = \dfrac{\pi^2 EI}{L^2}$ $P \propto I$ or $P \propto d^4$ or $\dfrac{p - p'}{p} = \dfrac{d^4 - (d^4)'}{d^4} = 1 - \left(\dfrac{0.8d}{d}\right)^4 = 0.59$

IES-19. A long slender bar having uniform rectangular cross-section 'B x H' is acted upon by an axial compressive force. The sides B and H are parallel to x- and y-axes respectively. The ends of the bar are fixed such that they behave as pin-jointed when the bar buckles in a plane normal to x-axis, and they behave as built-in when the bar buckles in a plane normal to y-axis. If load capacity in either mode of buckling is same, then the value of H/B will be: [IES-2000]

(a) 2 (b) 4 (c) 8 (d) 16

IES-19. Ans. (a) $P_{xx} = \dfrac{\pi^2 EI}{L^2}$ and $P_{yy} = \dfrac{4\pi^2 EI'}{L^2}$ as $P_{xx} = P_{yy}$ then $I = 4I'$ or $\dfrac{BH^3}{12} = 4 \times \dfrac{HB^3}{12}$ or $\dfrac{H}{B} = 2$

IES-20. The Euler's crippling load for a 2m long slender steel rod of uniform cross-section hinged at both the ends is 1 kN. The Euler's crippling load for 1 m long steel rod of the same cross-section and hinged at both ends will be: [IES-1998]

(a) 0.25 kN (b) 0.5 kN (c) 2 kN (d) 4 kN

IES-20. Ans. (d) For column with both ends hinged, $P = \dfrac{\pi^2 EI}{l^2}$. If 'l' is halved, P will be 4 times.

IES-21. If σ_c and E denote the crushing stress and Young's modulus for the material of a column, then the Euler formula can be applied for determination of cripping load of a column made of this material only, if its slenderness ratio is:

(a) More than $\pi\sqrt{E/\sigma_c}$ (b) Less than $\pi\sqrt{E/\sigma_c}$ **[IES-2005]**

(c) More than $\pi^2\left(\frac{E}{\sigma_c}\right)$ (d) Less than $\pi^2\left(\frac{E}{\sigma_c}\right)$

IES-21. Ans. (a) For long column $P_{Euler} < P_{crushing}$

or $\frac{\pi^2 EI}{l_e^2} < \sigma_c A$ or $\frac{\pi^2 EAK^2}{l_e^2} < \sigma_c A$ or $\left(\frac{le}{k}\right)^2 > \frac{\pi^2 E}{\sigma_c}$ or $\frac{le}{k} > \pi\sqrt{E/\sigma_c}$

IES-22. Four vertical columns of same material, height and weight have the same end conditions. Which cross-section will carry the maximum load? [IES-2009]

(a) Solid circular section (b) Thin hollow circular section
(c) Solid square section (d) I-section

IES-22. Ans. (b)

Rankine's Hypothesis for Struts/Columns

IES-23. Rankine Gordon formula for buckling is valid for [IES-1994]

(a) Long column (b) Short column
(c) Short and long column (d) Very long column

IES-23. Ans. (c)

Prof. Perry's formula

IES-24. Match List-I with List-II and select the correct answer using the code given below the lists: [IES-2008]

List-I (Formula/theorem/ method)	**List-II (Deals with topic)**
A. Clapeyron's theorem	**1.** Deflection of beam
B. Maculay's method	**2.** Eccentrically loaded column
C. Perry's formula	**3.** Riveted joints
	4. Continuous beam

Code:	A	B	C		A	B	C
(a)	3	2	1	(b)	4	1	2
(c)	4	1	3	(d)	2	4	3

IES-24. Ans. (b)

Previous 20-Years IAS Questions

Classification of Column

IAS-1. Mach List-I with List-II and select the correct answer using the codes given below the lists: [IAS-1999]

List-I	**List-II**
A. Polar moment of inertia of section	**1.** Thin cylindrical shell
B. Buckling	**2.** Torsion of shafts
C. Neutral axis	**3.** Columns
D. Hoop stress	**4.** Bending of beams

Codes:	A	B	C	D		A	B	C	D
(a)	3	2	1	4	(b)	2	3	4	1
(c)	3	2	4	1	(d)	2	3	1	4

IAS-1. Ans. (b)

Strength of Column

IAS-2. Assertion (A): A long column of square cross-section has greater buckling stability than that of a column of circular cross-section of same length, same material, same end conditions and same area of cross-section. [IAS-1998]

Reason (R): The second moment of area of a column of circular cross-section is smaller than that of a column of square cross section having the same area.

(a) Both A and R are individually true and R is the correct explanation of A

(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-2. Ans. (a)

IAS-3. Which one of the following pairs is *not* correctly matched? [IAS-2003]

(a) Slenderness ratio : The ratio of length of the column to the least radius of gyration
(b) Buckling factor : The ratio of maximum load to the permissible axial load on the column
(c) Short column : A column for which slenderness ratio < 32
(d) Strut : A member of a structure in any position and carrying an axial compressive load

IAS-3. Ans. (b) Buckling factor: The ratio of equivalent length of the column to the least radius of gyration.

Equivalent Length

IAS-4. A column of length 'I' is fixed at its both ends. The equivalent length of the column is: [IAS-1995]

(a) $2\,l$ (b) $0.5\,l$ (c) $2\,l$ (d) l

IAS-4. Ans. (b)

IAS-5. Which one of the following statements is correct? [IAS-2000]

(a) Euler's formula holds good only for short columns
(b) A short column is one which has the ratio of its length to least radius of gyration greater than 100
(c) A column with both ends fixed has minimum equivalent or effective length
(d) The equivalent length of a column with one end fixed and other end hinged is half of its actual length

IAS-5. Ans. (c) A column with both ends fixed has minimum equivalent effective length (l/2)

Euler's Theory (For long column)

IAS-6. For which one of the following columns, Euler buckling load = $\frac{4\pi^2 EI}{l^2}$?

(a) Column with both hinged ends **[IAS-1999; 2004]**
(b) Column with one end fixed and other end free
(c) Column with both ends fixed
(d) Column with one end fixed and other hinged

IAS-6. Ans. (c)

IAS-7. Assertion (A): Buckling of long columns causes plastic deformation. [IAS-2001]
Reason (R): In a buckled column, the stresses do not exceed the yield stress.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IAS-7. Ans. (d) And Critical Load for one end fixed, and other end free = $\pi^2 EI/4l^2$

IAS-8. Match List-I with List-II and select the correct answer using the code given below the Lists: [IES-1995; 2007; IAS-1997]

List-I (Long Column)	**List-II (Critical Load)**
A. Both ends hinged	**1.** $\pi^2 EI/4l^2$
B. One end fixed, and other end free	**2.** $4\pi^2 EI/l^2$

C. Both ends fixed	**3.** $2\pi^2EI/l^2$
D. One end fixed, and other end hinged	**4.** π^2EI/l^2

Code:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	2	1	4	3	(b)	4	1	2	3
(c)	2	3	4	1	(d)	4	3	2	1

IAS-8. Ans. (b)

Previous Conventional Questions with Answers

Conventional Question ESE-2001, ESE 2000

***Question*:** **Differentiate between strut and column. What is the general expression used for determining of their critical load?**

***Answer*:** **Strut:** A member of structure which carries an axial compressive load.

***Column*:** If the strut is vertical it is known as column.

For strut failure due to compression or $\sigma_c = \dfrac{Compressive\ \text{force}}{Area}$

If $\sigma_c > \sigma_{yc}$ it fails.

Euler's formula for column $(P_C) = \dfrac{\pi^2 EI}{\ell_e^2}$

Conventional Question ESE-2009

Q. **Two long columns are made of identical lengths '*l*' and flexural rigidities 'EI'. Column 1 is hinged at both ends whereas for column 2 one end is fixed and the other end is free.**

(i) Write the expression for Euler's buckling load for column 1.

(ii) What is the ratio of Euler's buckling load of column 1 to that column 2? [2 Marks]

Ans. (i)
$$\mathbf{P_1 = \frac{\pi^2 EI}{L^2};\ P_2 = \frac{\pi^2 EI}{4L^2}(right)}$$

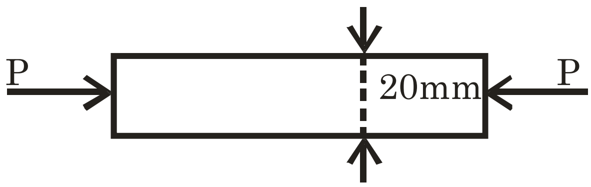
$$\textbf{For column 1, both end hinged } \boxed{\mathbf{l_e = L}}$$

(ii)
$$\mathbf{\frac{P_1}{P_2} = 4}$$

Conventional Question ESE-2010

Q. **The piston rod of diameter 20 mm and length 700 mm in a hydraulic cylinder is subjected to a compressive force of 10 kN due to internal pressure. The piston end of the rod is guided along the cylinder and the other end of the rod is hinged at the cross-head. The modulus of elasticity for piston rod material is 200 GPa. Estimate the factor of safety taken for the piston rod design. [2 Marks]**

Ans.



$\mathbf{\sigma = \frac{P}{A}}$; $\mathbf{\delta = \frac{PL}{AE}}$; $\ell_e = \dfrac{\ell}{\sqrt{2}}$; $\mathbf{P_e = \frac{\pi^2 EI}{\ell_e^2}}$ (considering one end of the column is fixed and other end is hinged)

Pe = Euler Crippling load

Compressive load, $\mathbf{P_c = \sigma_c \times Area}$ = 10 kN

Euler's load, $\mathbf{P_e} = \dfrac{\mathbf{2\pi^2 \times (200\times 10^9) \times (\pi \times 0.020^4 / 64)}}{\mathbf{(0.7)^2}}$ = 63.278 kN

$$\text{F.S} = \frac{\textbf{Euler's load}}{\textbf{Compressive load}}$$

$$\text{F.S} = \mathbf{\frac{63.278}{10} = 6.3}$$

Conventional Question ESE-1999

Question: **State the limitation of Euler's formula for calculating critical load on columns**

Answer: **Assumptions:**

(i) The column is perfectly straight and of uniform cross-section
(ii) The material is homogenous and isotropic
(iii) The material behaves elastically
(iv) The load is perfectly axial and passes through the centroid of the column section.
(v) The weight of the column is neglected.

Conventional Question ESE-2007

Question: **What is the value of Euler's buckling load for an axially loaded pin-ended (hinged at both ends) strut of length 'l' and flexural rigidity 'EI'? What would be order of Euler's buckling load carrying capacity of a similar strut but fixed at both ends in terms of the load carrying capacity of the earlier one?**

Answer: From Euler's buckling load formula,

$$\text{Critical load } (P_C) = \frac{\pi^2 EI}{\ell_e^2}$$

Equivalent length $(\ell_e) = \ell$ for both end hinged = $\ell/2$ for both end fixed.

$$\text{So for both end hinged } (P_c)_{beh} = \frac{\pi^2 EI}{\ell^2}$$

$$\text{and for both fixed } (P_c)_{bef} = \frac{\pi^2 EI}{\left(\ell/2\right)^2} = \frac{4\pi^2 EI}{\ell^2}$$

Conventional Question ESE-1996

Question: **Euler's critical load for a column with both ends hinged is found as 40 kN. What would be the change in the critical load if both ends are fixed?**

Answer: We know that Euler's critical laod,

$P_{Euler} = \dfrac{\pi^2 EI}{\ell_e^2}$ [Where E = modulus of elasticity, I = least moment of inertia ℓ_e = equivalent length]

For both end hinged $(\ell_e) = \ell$
And For both end fixed $(\ell_e) = \ell/2$

$$\therefore (P_{Euler})_{b.e.h.} = \frac{\pi^2 EI}{\ell^2} = 40\,\text{kN (Given)}$$

$$\text{and } (P_{Euler})_{b.e.F.} = \frac{\pi^2 EI}{(\ell/2)^2} = 4 \times \frac{\pi^2 EI}{\ell^2} = 4 \times 40 = 160\,kN$$

Conventional Question ESE-1999

Question: **A hollow cast iron column of 300 mm external diameter and 220 mm internal diameter is used as a column 4 m long with both ends hinged. Determine the safe compressive load the column can carry without buckling using Euler's formula and Rankine's formula**
E = 0.7×10^5 N/mm^2, FOS = 4, Rankine constant (a) = 1/1600
Crushing Stress (σ_c) = 567 N/mm^2

Answer: Given outer diameter of column (D) = 300 mm = 0.3 m.
Inner diameter of the column (d) = 220 mm = 0.22 m.
Length of the column (ℓ) = 4 m
End conditions is both ends hinged. Therefore equivalent length (ℓ_e) = ℓ = 4 m.

Yield crushing stress (σ_c) = 567 MPa = 567×10^6 N/m^2

Rankine constant (a) = $1/1600$ and E = 0.7×10^5 N/mm^2 = 70 x 10^9 N/m^2

$$\text{Moment of Inertia (I)} = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}\left[0.3^4 - 0.22^4\right] = 2.826\times10^{-4}\,m^4$$

$$\text{Slenderness ratio(k)}=\sqrt{\frac{I}{A}} = \sqrt{\frac{\frac{\pi}{64}\left(D^4 - d^4\right)}{\frac{\pi}{4}(D^2 - d^2)}} = \sqrt{\frac{D^2 + d^2}{16}} = \sqrt{\frac{0.3^2 + 0.22^2}{16}} = 0.093m$$

$$\text{Area(A)} = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(0.3^2 - 0.22^2) = 0.03267\,m^2$$

(i) Euler's buckling load, P_{Euler}

$$P_{Euler} = \frac{\pi^2 EI}{\ell_e^2} = \frac{\pi^2\times(70\times10^9)\times(2.826\times10^{-4})}{4^2} = 12.2\text{MN}$$

$$\therefore \text{Safe load} = \frac{P_{Euler}}{\text{fos}} = \frac{12.2}{4} = 3.05\,MN$$

(ii)Rankine's buckling load, $P_{Rankine}$

$$P_{Rankine} = \frac{\sigma_c.A}{1 + a.\left(\frac{\ell_e}{k}\right)^2} = \frac{\left(567\times10^6\right)\times0.03267}{1+\frac{1}{1600}\times\left(\frac{4}{0.093}\right)^2} = 8.59\text{ MN}$$

$$\therefore \textit{Safe}\text{ load} = \frac{P_{Rankine}}{\text{fos}} = \frac{8.59}{4} = 2.148\,MPa$$

Conventional Question ESE-2008

***Question*:** **A both ends hinged cast iron hollow cylindrical column 3 m in length has a critical buckling load of P kN. When the column is fixed at both the ends, its critical buckling load raise by 300 kN more. If ratio of external diameter to internal diameter is 1.25 and E = 100 GPa determine the external diameter of column.**

***Answer*:** $$P_c = \frac{\pi^2 EI}{I_e^2}$$

For both end hinged column

$$P=\frac{\pi^2\text{EI}}{\text{L}^2} --- (i)$$

For both end fixed column

$$P+300=\frac{\pi^2 EI}{\left(L/2\right)^2} = \frac{4\pi^2 EI}{L^2} --- (ii)$$

Dividing (ii) by (i) we get

$$\frac{P+300}{P} = 4 \text{ or } P=100\text{kN}$$

Moment of inertia of a hollow cylinder c/s is

$$I = \frac{\pi}{64}\left(D^4 - d^4\right) = \frac{PL^2}{\pi^2 E}$$

$$or D^4 - d^4 = \frac{64}{\pi}\frac{\left(100\times10^3\right)3^2}{\pi^2\times100\times10^9} = 1.8577\times10^{-5}$$

$$given\ \frac{D}{d} = 1.25 \text{ or d} = \frac{D}{1.25\text{-}5}$$

$$or\ D^4\left[1-\left(\frac{1}{1.25}\right)^4\right] = 1.8577\times10^{-5}$$

$$or\ D=0.0749 \text{ m} = 74.9 \text{ mm}$$

Conventional Question AMIE-1996

Question: **A piston rod of steam engine 80 cm long in subjected to a maximum load of 60 kN. Determine the diameter of the rod using Rankine's formula with permissible compressive stress of 100 N/mm². Take constant in Rankine's formula as $\frac{1}{7500}$ for hinged ends. The rod may be assumed partially fixed with length coefficient of 0·6.**

Answer: Given: $l = 80 \text{ cm} = 800\text{mm}; P = 60\text{kN} = 60\times10^3\text{N}, \sigma_c = 100\text{N/mm}^2;$

$a = \frac{1}{7500}$ for hinged ends; length coefficient $= 0.6$

To find diameter of the rod, d:

Use Rankine's formula

$$P = \frac{\sigma_c A}{1+a\left(\frac{l_e}{k}\right)^2}$$

Here $l_e = 0.6l = 0.6\times800 = 480$ mm [$\because$length coefficient$=0.6$]

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{\frac{\pi}{64}d^4}{\frac{\pi}{4}d^2}} = \frac{d}{4}$$

$$\therefore \qquad 60\times10^3 = \frac{100\times\left(\frac{\pi}{4}d^2\right)}{1+\frac{1}{7500}\left[\frac{480}{d/4}\right]^2}$$

Solving the above equation we get the value of 'd'

Note: Unit of d comes out from the equation will be mm as we put the equivalent length in mm.

or $\quad d = 33.23\text{mm}$

Conventional Question ESE-2005

Question: **A hollow cylinder CI column, 3 m long its internal and external diameters as 80 mm and 100 mm respectively. Calculate the safe load using Rankine formula: if**

(i) Both ends are hinged and

(ii) Both ends are fixed.

Take crushing strength of material as 600 N/mm^2, Rankine constant 1/1600 and factor of safety = 3.

Answer: Moment of Inertia (I)$=\frac{\pi}{64}(0.1^4 - 0.08^4)m^4 = 2.898\times10^{-6}m^4$

$$\text{Area}(A) = \frac{\pi}{4}\left(0.1^2 - 0.08^2\right) = 2.8274\times10^{-3} m^2$$

$$\text{Radius of gyration (k)} = \sqrt{\frac{I}{A}} = \sqrt{\frac{2.898\times10^{-6}}{2.8274\times10^{-3}}} = 0.032m$$

$$P_{Rankine} = \frac{\sigma_c.A}{1+a\left(\frac{\ell_e}{k}\right)^2}; \quad [\ell_e = \text{equivalent length}]$$

(i) $$= \frac{\left(600\times10^6\right)\times\left(2.8274\times10^{-3}\right)}{1+\left(\frac{1}{1600}\right)\times\left(\frac{3}{0.032}\right)^2}$$; [ℓ_e = l = 3 m for both end hinged]

$$=2.61026\text{kN}$$

$$\text{Safe load (P)} = \frac{P_{Rankine}}{FOS} = \frac{26126}{3} = 87.09kN$$

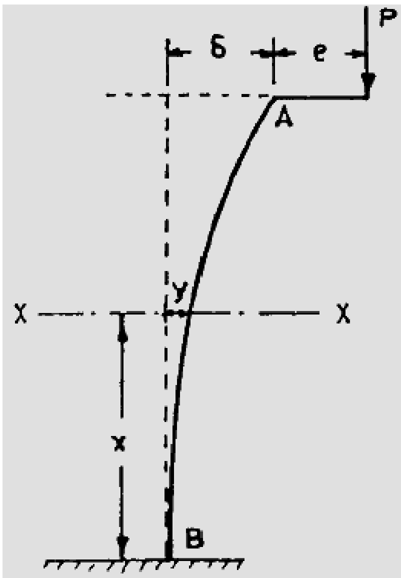
(ii) For both end fixed, $\ell_e = \ell/2 = 1.5m$

$$P_{Rankine} = \frac{\left(600\times10^6\right)\times(2.8274\times10^{-3})}{1+\frac{1}{1600}\times\left(\frac{1.5}{0.032}\right)^2} = 714.8kN$$

$$\textit{Safe}\text{ load (P)} = \frac{P_{Rankine}}{FOS} = \frac{714.8}{3} = 238.27kN$$

Conventional Question AMIE-1997

***Question*:** **A slender column is built-in at one end and an eccentric load is applied at the free end. Working from the first principles find the expression for the maximum length of column such that the deflection of the free end does not exceed the eccentricity of loading.**



***Answer*:** Above figure shows a slender column of length 'I'. The column is built in at one end B and eccentric load P is applied at the free end A.

Let y be the deflection at any section XX distant x from the fixed end B. Let δ be the deflection at A.

The bending moment at the section XX is given by

$$EI\frac{d^2y}{dx^2} = P(\delta + e - y) \qquad ----(i)$$

$$EI\frac{d^2y}{dx^2} + Py = P(\delta + e) \quad \text{or} \quad \frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{P}{EI}(\delta + e)$$

The solution to the above differential equation is

$$y = C_1\cos\left[x\sqrt{\frac{P}{EI}}\right] + C_2\sin\left[x\sqrt{\frac{P}{EI}}\right] + (\delta + e) \qquad ---(ii)$$

Where C_1 and C_2 are the constants.

At the end B, $x = 0$ and $y = 0$

$$\therefore \qquad 0 = C_1\cos 0 + C_2\sin 0 + (\delta + e)$$

$$\text{or} \qquad C_1 = -(\delta + e)$$

Differentiating equation (ii) we get

$$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\sin\left[x\sqrt{\frac{P}{EI}}\right] + C_2\sqrt{\frac{P}{EI}}\cos\left[x\sqrt{\frac{P}{EI}}\right]$$

Again, at the fixed end B,

When $x = 0, \frac{dy}{dx} = 0$

$$\therefore \qquad 0 = (\delta + e)\sqrt{\frac{P}{EI}} \times 0 + C_2\sqrt{\frac{P}{EI}}\cos 0$$

$$\text{or} \qquad C_2 = 0$$

At the free end A, $x = \ell, y = \delta$

Substituting for x and y in equation (ii), we have

$$\delta = -(\delta + e)\cos\left[\ell\sqrt{\frac{P}{EI}}\right] = (\delta + e)$$

$$\therefore \qquad \cos\left[\ell\sqrt{\frac{P}{EI}}\right] = \frac{e}{\delta + e} \qquad ---(iii)$$

It is mentioned in the problem that the deflection of the free end does not exceed the eccentricity. It means that $\delta = e$

Substituting this value in equation (iii), we have

$$\cos\left[\ell\sqrt{\frac{P}{EI}}\right] = \frac{e}{\delta + e} = \frac{1}{2}$$

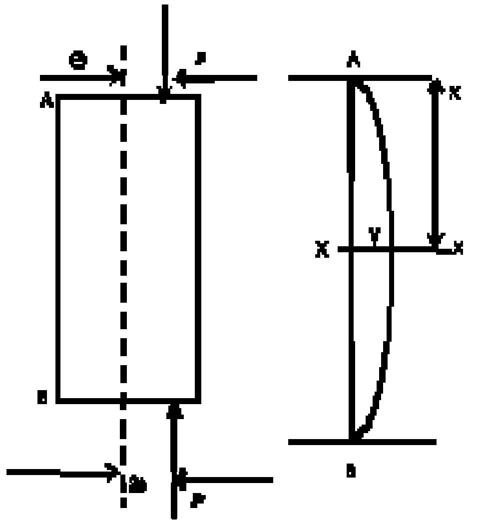
$$\therefore \qquad \ell\sqrt{\frac{P}{EI}} = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\therefore \qquad \ell = \frac{\pi}{3}\sqrt{\frac{EI}{P}}$$

Conventional Question ESE-2005

***Question*:** **A long strut AB of length 'ℓ' is of uniform section throughout. A thrust P is applied at the ends eccentrically on the same side of the centre line with eccentricity at the end B twice than that at the end A. Show that the maximum bending moment occurs at a distance x from the end A,**

Where, tan(kx)= $\frac{2 - \cos k\ell}{\sin k\ell}$ and $k=\sqrt{\frac{P}{EI}}$



Answer: Let at a distance 'x' from end A deflection of the beam is y

$$\therefore EI\frac{d^2y}{dx^2} = -P.y$$

$$or\ \frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

$$or\ \frac{d^2y}{dx^2} + k^2y = 0 \qquad \left[\because k = \sqrt{\frac{P}{EI}}\,given\right]$$

C.F of this differential equation

y = A cos kx + B sin kx, Where A & B constant.

It is clear at x = 0, y = e

And at x = ℓ, y= 2e

$$\therefore e = A.................(i)$$

$$2e = A\cos k\ell + B\sin k\ell \qquad or\quad B = \left[\frac{2e - e\cos k\ell}{\sin k\ell}\right]$$

$$\therefore y = e\cos kx + \left[\frac{2e - e\cos k\ell}{\sin k\ell}\right]\sin kx$$

Where bending moment is maximum,

the deflection will be maximum so $\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -ek\sin kx + k.\left[\frac{2e - e\cos k\ell}{\sin k\ell}\right]\cos kx = 0$$

$$or\ \tan kx = \frac{2 - \cos k\ell}{\sin k\ell}$$

Conventional Question ESE-1996

Question: **The link of a mechanism is subjected to axial compressive force. It has solid circular cross-section with diameter 9 mm and length 200 mm. The two ends of the link are hinged. It is made of steel having yield strength = 400 N/mm² and elastic modulus = 200 kN/mm². Calculate the critical load that the link can carry. Use Johnon's equation.**

Answer: According to Johnson's equation

$$P_{cr} = \sigma_y.A\left[1 - \frac{\sigma_y}{4n\pi^2E}\left(\frac{\ell}{k}\right)^2\right]$$

Hear A=area of cross section= $\frac{\pi d^2}{4} = 63.62\,mm^2$

least radius of gyration (k) $=\sqrt{\frac{I}{A}} = \sqrt{\frac{\left(\frac{\pi d^4}{64}\right)}{\left(\frac{\pi d^2}{4}\right)}} = \frac{d}{4} = 2.55\,mm$

For both end hinged n=1

$$\therefore P_{cr} = 400\times 63.62\left[1 - \frac{400}{4\times1\times\pi^2\times(200\times10^3)\times}\left(\frac{200}{2.25}\right)^2\right] = 15.262\,kN$$

Conventional Question GATE-1995

Question: **Find the shortest length of a hinged steel column having a rectangular cross-section 600 mm × 100 mm, for which the elastic Euler formula applies. Take**

yield strength and modulus of elasticity value for steel as 250 MPa and 200 GPa respectively.

Answer: Given: Cross-section, (= b x d) = 600 mm x 100 mm = 0.6 m x 0.1 m = 0.06 m^2;

Yield strength = $\frac{P}{A} = 250MPa = 250MN/m^2; E = 200\ GPa = 200 \times 10^{12} N/m^2$

Length of the column, L :

Least area moment of Inertia, $I = \frac{bd^3}{12} = \frac{0.6 \times 0.1^3}{12} = 5 \times 10^{-5} \text{m}^4$

Also, $k^2 = \frac{I}{A} = \frac{5 \times 10^{-5}}{0.6 \times 0.1} = 8.333 \times 10^{-4}\ m^2$

[$\because I = AK^2$ (where $A =$ area of cross-section, $k =$ radius of gyration)]

From Euler's formula for column, we have

Crushing load, $P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{L^2}$

For both end hinged type of column, L_e = L

or $P_{cr} = \frac{\pi^2 EAk^2}{L^2}$

or $Yield\ stress\left(\frac{P_{cr}}{A}\right) = \frac{\pi^2 EI}{L^2}$

or $L^2 = \frac{\pi^2 Ek^2}{(P_{cr}/A)}$

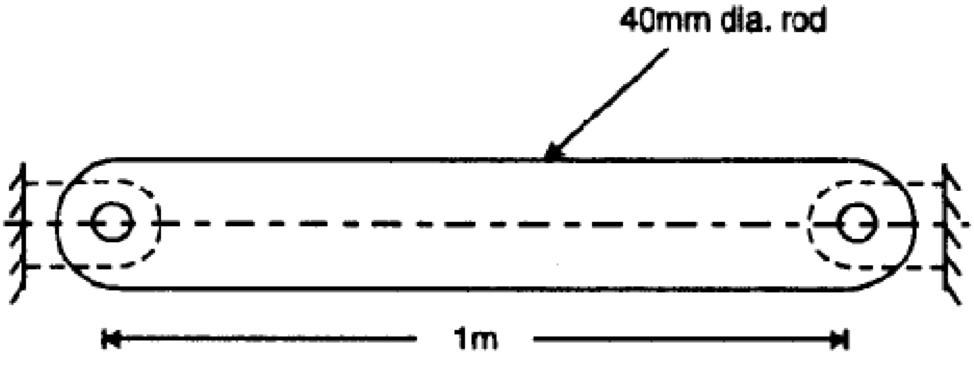
Substituting the value, we *get*

$L^2 = \frac{\pi^2 \times 200 \times 10^9 \times 0.0008333}{250 \times 10^6} = 6.58$

$L = 2.565\, m$

Conventional Question GATE-1993

Question: **Determine the temperature rise necessary to induce buckling in a lm long circular rod of diameter 40 mm shown in the Figure below. Assume the rod to be pinned at its ends and the coefficient of thermal expansion as** $20 \times 10^{-6} /^0 C$ **. Assume uniform heating of the bar.**



Answer: Let us assume the buckling load be'P'.

$\delta L = L.\propto.\triangle t,$ Where $\triangle t$ is the temperature rise.

or $\triangle t = \dfrac{\delta L}{L.\propto}$

Also, $\delta L = \dfrac{PL}{AE}$ *or* $P = \dfrac{\delta L.AE}{L}$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \qquad ---\left(\text{where } L_e = \text{equivalent length}\right)$$

or $\dfrac{\pi^2 EI}{L^2} = \dfrac{\delta L.A.E}{L}$ $\left[\because L_e = L \text{ For both end hinged}\right]$

or $\delta L = \dfrac{\pi^2 I}{LA}$

$$\triangle t = \frac{\delta L}{L.\propto} = \frac{\pi^2 I}{LA.L.\propto} = \frac{\pi^2 I}{L^2 A.\propto}$$

Substituting the values, we get

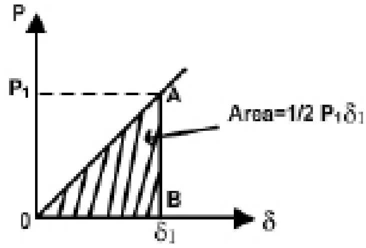
Temperature rise $\triangle t = \dfrac{\pi^2 \times \dfrac{\pi}{64} \times (0.040)^4}{(1)^2 \times \dfrac{\pi}{4} \times (0.040)^4 \times 20 \times 10^{-6}} = 49.35^0 C$

So the rod will buckle when the temperature rises more than 49.35°C.

14. Strain Energy Method

Theory at a Glance (for IES, GATE, PSU)

1. Resilience (U)



- Resilience is an ability of a material to absorb energy when elastically deformed and to return it when unloaded.
- The strain energy stored in a specimen when stained ***within*** the elastic limit is known as resilience.

$$\boxed{U = \frac{\sigma^2}{2E} \times Volume} \quad or \quad \boxed{U = \frac{\in^2 E}{2} \times Volume}$$

2. Proof Resilience

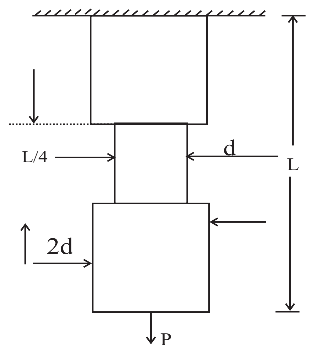
- Maximum strain energy stored at elastic limit. i.e. the strain energy stored in the body ***upto*** elastic limit.
- This is the property of the material that enables it to resist shock and impact by storing energy. The measure of proof resilience is the strain energy absorbed per unit volume.

3. Modulus of Resilience (u)

The proof resilience per unit volume is known as modulus of resilience. If σ is the stress due to gradually applied load, then

$$\boxed{u = \frac{\sigma^2}{2E}} \quad or \quad \boxed{u = \frac{\in^2 E}{2}}$$

4. Application



$$U = \frac{P^2L}{2AE} = \frac{P^2 \frac{3}{4}L}{2\frac{\pi}{4}(2d)^2E} + \frac{P^2 . \frac{L}{4}}{2.\frac{\pi d^2}{4}E}$$

Strain energy becomes smaller & smaller as the cross sectional area of bar is increased over more & more of its length i.e. A↑, U↓

5. Toughness

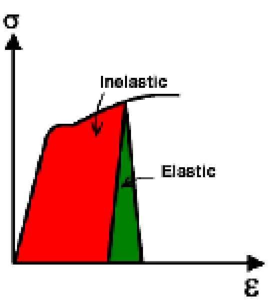
- This is the property which enables a material to be twisted, bent or stretched under impact load or high stress before rupture. It may be considered to be the ability of the material to

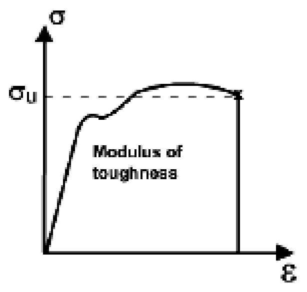
absorb energy in the plastic zone. The measure of toughness is the amount of energy absorbed after being stressed upto the point of fracture.

- Toughness is an ability to absorb energy in the plastic range.
- The ability to withstand occasional stresses above the yield stress without fracture.
- Toughness = strength + ductility
- The materials with higher modulus of toughness are used to make components and structures that will be exposed to sudden and impact loads.

Modulus of Toughness

- The ability of unit volume of material to absorb energy in the plastic range.
- The amount of work per unit volume that the material can withstand without failure.
- The area under the entire stress strain diagram is called *modulus of toughness*, which is a measure of energy that can be absorbed by the unit volume of material due to impact loading before it fractures.



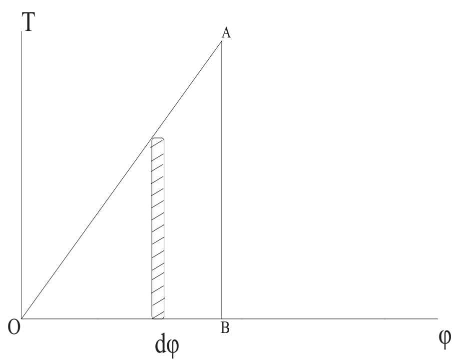


$U_T = \sigma_u \varepsilon_f$

6. Strain energy in shear and torsion

- Strain energy per unit volume, (u_s)

$$\boxed{u_s = \frac{\tau^2}{2G}} \text{ or, } \boxed{u_s = \frac{G\gamma^2}{2}}$$



- Total Strain Energy (U) for a Shaft in Torsion

$$U_s = \frac{1}{2}T\phi$$

$$\boxed{\therefore U_s = \frac{1}{2}\left(\frac{T^2L}{GJ}\right)} \quad or \quad \frac{1}{2}\frac{GJ\phi^2}{L}$$

$$\text{or} \quad \boxed{U_s = \frac{\tau_{max}^2}{2G}\frac{2\pi L}{r^2}\int \rho^2 d\rho}$$

- **Cases**

$$\bullet Solid\ shaft, U_s = \frac{\tau_{max}^2}{4G} \times \pi r^2 L$$

$$\bullet Hollow\ shaft,\ U_s = \frac{\tau_{max}^2}{4G} \times \frac{\pi\left(D^4 - d^4\right)L}{D^2} = \frac{\tau_{max}^2}{4G} \times \frac{\left(D^2 + d^2\right)}{D^2} \times Volume$$

$\bullet$ *Thin walled tube*, $U_s = \dfrac{\tau^2}{4G} \times sLt$

where s = Length of mean centre line

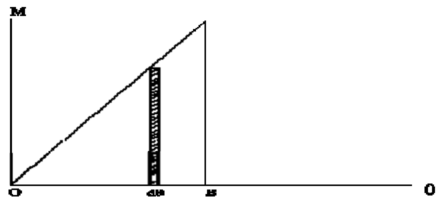
$\bullet$ *Conical spring*, $U_S = \dfrac{GJ}{2}\int\left(\dfrac{d\phi}{dx}\right)^2 dx = \dfrac{GJ}{2}\int_0^{2\pi n}\left(\dfrac{PR}{GJ}\right)^2 .R.d\alpha \quad (R = Radius)$

$$= \frac{P^2}{2GJ}\int_0^{2\pi n} R^3 d\alpha \quad (R \text{ varies with } \propto)$$

$\bullet$ Cantilever beam with load 'p' at end, $U_s = \dfrac{3}{5}\left(\dfrac{P^2L}{bhG}\right)$

$\bullet$ *Helical spring*, $U_s = \dfrac{\pi P^2 R^3 n}{GJ} \qquad (\because L = 2\pi Rn)$

7. Strain energy in bending.



- Angle subtended by arc, $\theta = \int \dfrac{M_x}{EI}.dx$
- Strain energy stored in beam.

$$U_b = \int_0^L \frac{M_x^2}{2EI}.dx$$

$$\text{or} \quad U_b = \frac{EI}{2}\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx \qquad \left(\because \frac{d^2y}{dx^2} = -\frac{M}{EI}\right)$$

- **Cases**
  - Cantilever beam with a end load P , $U_b = \dfrac{P^2L^3}{6EI}$
  - Simply supported with a load P at centre, $U_b = \dfrac{P^2L^3}{96EI}$
- **Important Note**
  - For pure bending
    - M is constant along the length 'L'
    - $\theta = \dfrac{ML}{EI}$
    - $U = \dfrac{M^2L}{2EI}$ if M is known $= \dfrac{EI\theta^2}{2L}$ *if curvature* θ / L is known
  - For non-uniform bending
    - Strain energy in shear is neglected
    - Strain energy in bending is only considered.

8. Castiglione's theorem

$$\frac{\partial U}{\partial P_n} = \delta_n$$

$$\frac{\partial U}{\partial p} = \frac{1}{EI}\int M_x\left(\frac{\partial M_x}{\partial p}\right)dx$$

- **Note:**
  - Strain energy, stored due to direct stress in 3 coordinates

$$U = \frac{1}{2E}\left[\sum(\sigma_x)^2 - 2\mu\sum\sigma_x\sigma_y\right]$$

  - If $\sigma_x = \sigma_y = \sigma_z$, in case of equal stress in 3 direction then

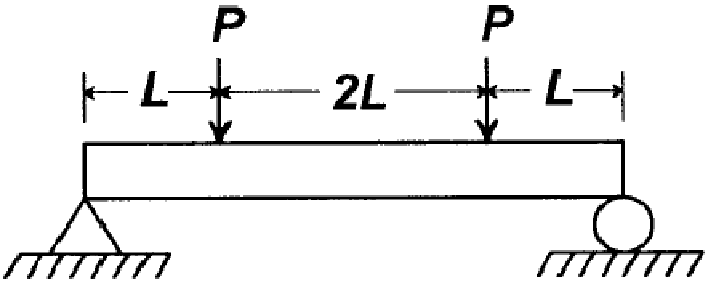
$$U = \frac{3\sigma^2}{2E}[1-2\mu] = \frac{\sigma^2}{2k} \quad \text{(volume strain energy)}$$

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Strain Energy or Resilience

GATE-1. The strain energy stored in the beam with flexural rigidity EI and loaded as shown in the figure is: [GATE-2008]



(a) $\dfrac{P^2L^3}{3EI}$ (b) $\dfrac{2P^2L^3}{3EI}$ (c) $\dfrac{4P^2L^3}{3EI}$ (d) $\dfrac{8P^2L^3}{3EI}$

GATE-1. Ans. (c) $\displaystyle\int_0^{4L}\frac{M^2dx}{EI}=\int_0^{L}\frac{M^2dx}{EI}+\int_L^{3L}\frac{M^2dx}{EI}+\int_{3L}^{4L}\frac{M^2dx}{EI}$

$$=2\int_0^{L}\frac{M^2dx}{EI}+\int_L^{3L}\frac{M^2dx}{EI}\qquad\left[\text{By symmetry}\int_0^{L}\frac{M^2dx}{EI}=\int_{3L}^{4L}\frac{M^2dx}{EI}\right]$$

$$=2\int_0^{L}\frac{(Px)^2dx}{EI}+\int_L^{3L}\frac{(PL)^2dx}{EI}=\frac{4P^2L^3}{3EI}$$

GATE-2. $\dfrac{PL^3}{3EI}$ is the deflection under the load P of a cantilever beam [length L, modulus of elasticity, E, moment of inertia-I]. The strain energy due to bending is: [GATE-1993]

$(a)\dfrac{P^2L^3}{3EI}$ $(b)\dfrac{P^2L^3}{6EI}$ $(c)\dfrac{P^2L^3}{4EI}$ $(d)\dfrac{P^2L^3}{48EI}$

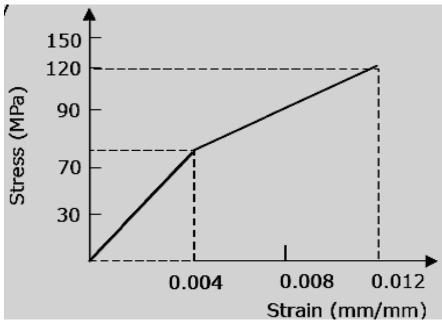
GATE-2. Ans. (b) We may do it taking average

Strain energy = Average force x displacement = $\left(\dfrac{P}{2}\right)\times\dfrac{PL^3}{3EI}=\dfrac{P^2L^3}{6EI}$

Alternative method: In a funny way you may use Castiglione's theorem, $\delta=\dfrac{\partial U}{\partial P}$. Then $\delta=\dfrac{\partial U}{\partial P}=\dfrac{PL^3}{3EI}$ or $U=\int\partial U=\int\dfrac{PL^3}{3EI}\partial P$ Partially integrating with respect to P we get $U=\dfrac{P^2L^3}{6EI}$

GATE-3. The stress-strain behaviour of a material is shown in figure. Its resilience and toughness, in Nm/m³, are respectively

(a) 28×10^4, 76×10^4
(b) 28×10^4, 48×10^4
(c) 14×10^4, 90×10^4
(d) 76×10^4



[GATE-2000]

GATE-3. Ans. (c) Resilience = area under this curve up to 0.004 strain

$$= \frac{1}{2} \times 0.004 \times 70 \times 10^6 = 14 \times 10^4 \text{ Nm/m}^3$$

Toughness = area under this curve up to 0.012 strain

$$= 14 \times 10^4 + 70 \times 10^6 \times (0.012 - 0.004) + \frac{1}{2} \times (0.012 - 0.004) \times (120 - 70) \times 10^6 \text{ Nm/m}^3$$

$$= 90 \times 10^4 \text{ Nm/m}^3$$

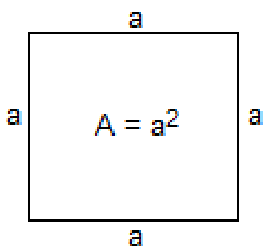
GATE-4. A square bar of side 4 cm and length 100 cm is subjected to an axial load P. The same bar is then used as a cantilever beam and subjected to all end load P. The ratio of the strain energies, stored in the bar in the second case to that stored in the first case, is: [GATE-1998]

(a) 16 (b) 400 (c) 1000 (d) 2500

GATE-4. Ans. (d) $U_1 = \dfrac{\left(\dfrac{W}{A}\right)^2 AL}{2E} = \dfrac{W^2 L}{2AE}$

$$U_2 = \frac{W^2 L^3}{6EI} = \frac{W^2 L^3}{6E\left(\frac{1}{12}a^4\right)} = \frac{2W^2 L^3}{Ea^4}$$

$$\text{or } \frac{U_2}{U_1} = \frac{4L^2}{a^2} = 4 \times \left(\frac{100}{4}\right)^2 = 2500$$



Toughness

GATE-5. The total area under the stress-strain curve of a mild steel specimen tested up to failure under tension is a measure of [GATE-2002]

(a) Ductility (b) Ultimate strength (c) Stiffness (d) Toughness

GATE-5. Ans. (d)

Previous 20-Years IES Questions

Strain Energy or Resilience

IES-1. What is the strain energy stored in a body of volume V with stress σ due to gradually applied load? [IES-2006]

(a) $\dfrac{\sigma E}{V}$ (b) $\dfrac{\sigma E^2}{V}$ (c) $\dfrac{\sigma V^2}{E}$ (d) $\dfrac{\sigma^2 V}{2E}$

Where, E = Modulus of elasticity

IES-1. Ans. (d) Strain Energy $= \dfrac{1}{2} . \dfrac{\sigma^2}{E} \times V$

IES-2. A bar having length L and uniform cross-section with area A is subjected to both tensile force P and torque T. If G is the shear modulus and E is the Young's modulus, the internal strain energy stored in the bar is: [IES-2003]

(a) $\dfrac{T^2L}{2GJ}+\dfrac{P^2L}{AE}$ (b) $\dfrac{T^2L}{GJ}+\dfrac{P^2L}{2AE}$ (c) $\dfrac{T^2L}{2GJ}+\dfrac{P^2L}{2AE}$ (d) $\dfrac{T^2L}{GJ}+\dfrac{P^2L}{AE}$

IES-2. Ans. (c) Internal strain energy $=\dfrac{1}{2}P\delta+\dfrac{1}{2}T\theta=\dfrac{1}{2}P\dfrac{PL}{AE}+\dfrac{1}{2}T\dfrac{TL}{GJ}$

IES-3. Strain energy stored in a body of volume V subjected to uniform stress *s* is: [IES-2002]

(a) s E / V (b) sE^2/V (c) sV^2/E (d) $s^2V/2E$

IES-3. Ans. (d)

IES-4. A bar of length L and of uniform cross-sectional area A and second moment of area 'I' is subjected to a pull P. If Young's modulus of elasticity of the bar material is E, the expression for strain energy stored in the bar will be: [IES-1999]

(a) $\dfrac{P^2L}{2AE}$ (b) $\dfrac{PL^2}{2EI}$ (c) $\dfrac{PL^2}{AE}$ (d) $\dfrac{P^2L}{AE}$

IES-4. Ans. (a) Strain energy $=\dfrac{1}{2}$ x stress x strain x volume $=\dfrac{1}{2}\times\left(\dfrac{P}{A}\right)\times\left(\dfrac{P}{A}.\dfrac{L}{E}\right)\times(AL)=\dfrac{PL^2}{2AE}$

IES-5. Which one of the following gives the correct expression for strain energy stored in a beam of length L and of uniform cross-section having moment of inertia 'I' and subjected to constant bending moment M? [IES-1997]

(a) $\dfrac{ML}{EI}$ (b) $\dfrac{ML}{2EI}$ (c) $\dfrac{M^2L}{EI}$ (d) $\dfrac{M^2L}{2EI}$

IES-5. Ans. (d)

IES-6. A steel specimen 150 mm^2 in cross-section stretches by 0·05 mm over a 50 mm gauge length under an axial load of 30 kN. What is the strain energy stored in the specimen? (Take E = 200 GPa) [IES-2009]

(a) 0.75 N-m (b) 1.00 N-m (c) 1.50 N-m (d) 3.00 N-m

IES-6. Ans. (a) Strain Energy stored in the specimen

$$=\frac{1}{2}P\delta=\frac{1}{2}P\left(\frac{PL}{AE}\right)=\frac{P^2L}{2AE}=\frac{(30000)^2\times50\times10^{-3}}{2\times150\times10^{-6}\times200\times10^{9}}=0.75\text{ N-m}$$

IES-7. What is the expression for the strain energy due to bending of a cantilever beam (length L. modulus of elasticity E and moment of inertia I)? [IES-2009]

(a) $\dfrac{P^2L^3}{3EI}$ (b) $\dfrac{P^2L^3}{6EI}$ (c) $\dfrac{P^2L^3}{4EI}$ (d) $\dfrac{P^2L^3}{48EI}$

IES-7. Ans. (b) Strain Energy Stored $=\displaystyle\int_0^L\frac{(Px)^2dx}{2E}=\frac{P^2}{2EI}\left(\frac{x^3}{3}\right)\Bigg|_0^L=\frac{P^2L^3}{6EI}$

IES-8. The property by which an amount of energy is absorbed by a material without plastic deformation, is called: [IES-2000]

(a) Toughness (b) Impact strength (c) Ductility (d) Resilience

IES-8. Ans. (d)

IES-9. 30 C 8 steel has its yield strength of 400 N/mm² and modulus of elasticity of 2 × 10⁵ MPa. Assuming the material to obey Hooke's law up to yielding, what is its proof resilience? [IES-2006]

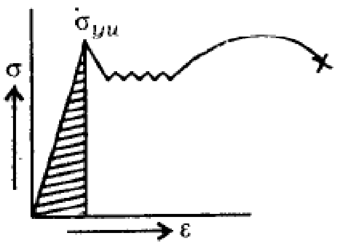
(a) 0·8 N/mm² (b) 0.4 N/mm² (c) 0·6 N/mm² (d) 0·7 N/mm²

IES-9. Ans. (b) Proof resilience $(R_p)=\dfrac{1}{2}.\dfrac{\sigma^2}{E}=\dfrac{1}{2}\times\dfrac{(400)^2}{2\times10^5}=0.4\,N/mm^2$

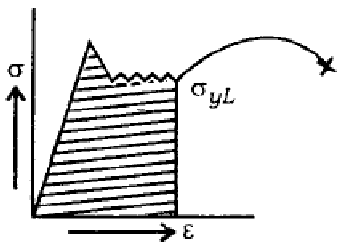
Toughness

IES-10. Toughness for mild steel under uni-axial tensile loading is given by the shaded portion of the stress-strain diagram as shown in [IES-2003]

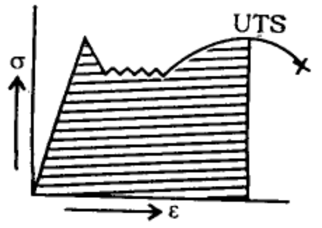
(a)



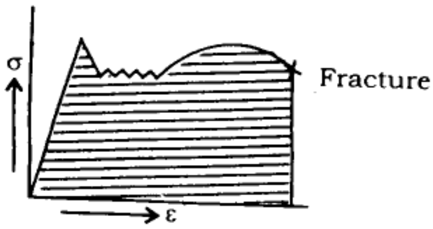
(b)



(c)



(d)



IES-10. Ans. (d) Toughness of material is the total area under stress-strain curve.

Previous 20-Years IAS Questions

Strain Energy or Resilience

IAS-1. Total strain energy stored in a simply supported beam of span, 'L' and flexural rigidity 'EI 'subjected to a concentrated load 'W' at the centre is equal to: [IAS-1995]

(a) $\dfrac{W^2L^3}{40EI}$ (b) $\dfrac{W^2L^3}{60EI}$ (c) $\dfrac{W^2L^3}{96EI}$ (d) $\dfrac{W^2L^3}{240EI}$

IAS-1. Ans. (c) Strain energy $= \int_0^L \dfrac{M^2dx}{2EI} = 2\times\int_0^{L/2}\dfrac{M^2dx}{2EI} = \dfrac{1}{EI}\times\int_0^{L/2}\left(\dfrac{Wx}{2}\right)^2 dx = \dfrac{W^2L^3}{96EI}$

Alternative method: In a funny way you may use Castiglione's theorem, $\delta = \dfrac{\partial U}{\partial P} = \dfrac{\partial U}{\partial W}$

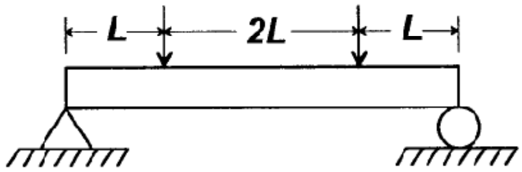
We know that $\delta = \dfrac{WL^3}{48EI}$ for simply supported beam in concentrated load at mid span. Then $\delta = \dfrac{\partial U}{\partial P} = \dfrac{\partial U}{\partial W} = \dfrac{WL^3}{48EI}$ or $U = \int \partial U = \int \dfrac{WL^3}{48EI}\partial W$ partially integrating with respect to W we get $U = \dfrac{W^2L^3}{96EI}$

IAS-2. If the cross-section of a member is subjected to a uniform shear stress of intensity 'q' then the strain energy stored per unit volume is equal to (G = modulus of rigidity). [IAS-1994]

(a) $2q^2/G$ (b) $2G / q^2$ (c) $q^2 /2G$ (d) $G/2 q^2$

IAS-2. Ans. (c)

IAS-3. The strain energy stored in the beam with flexural rigidity EI and loaded as shown in the figure is: [GATE-2008]



(a) $\dfrac{P^2L^3}{3EI}$ (b) $\dfrac{2P^2L^3}{3EI}$ (c) $\dfrac{4P^2L^3}{3EI}$ (d) $\dfrac{8P^2L^3}{3EI}$

IAS-3. Ans. (c) $\int_0^{4L}\frac{M^2dx}{EI}=\int_0^{L}\frac{M^2dx}{EI}+\int_L^{3L}\frac{M^2dx}{EI}+\int_{3L}^{4L}\frac{M^2dx}{EI}$

$$=2\int_0^{L}\frac{M^2dx}{EI}+\int_L^{3L}\frac{M^2dx}{EI}\qquad\left[\text{By symmetry}\int_0^{L}\frac{M^2dx}{EI}=\int_{3L}^{4L}\frac{M^2dx}{EI}\right]$$

$$=2\int_0^{L}\frac{(Px)^2dx}{EI}+\int_L^{3L}\frac{(PL)^2dx}{EI}=\frac{4P^2L^3}{3EI}$$

IAS-4. Which one of the following statements is correct? **[IAS-2004]**
The work done in stretching an elastic string varies
(a) As the square of the extension
(b) As the square root of the extension
(c) Linearly with the extension
(d) As the cube root of the extension

IAS-4. Ans. (a) $\dfrac{\sigma^2}{2E}=\dfrac{1}{2}\in^2 E=\dfrac{1}{2}\left[\dfrac{(\delta l)^2}{L^2}\right]E$

Toughness

IAS-5. Match List-I with List-II and select the correct answer using the codes given below the lists: **[IAS-1996]**

List-I (Mechanical properties)
A. Ductility
B. Hardness
C. Malleability
D. Toughness

List-II (Meaning of properties)
1. Resistance to indentation
2. Ability to absorb energy during plastic deformation
3. Percentage of elongation
4. Ability to be rolled into flat product

Codes:	A	B	C	D		A	B	C	D
(a)	1	4	3	2	(b)	3	2	4	1
(c)	2	3	4	1	(d)	3	1	4	2

IAS-5. Ans. (d)

IAS-6. Match List-I (Material properties) with List-II (Technical definition/requirement) and select the correct answer using the codes below the lists: **[IAS-1999]**

List-I
A. Hardness
B. Toughness
C. Malleability
D. Ductility

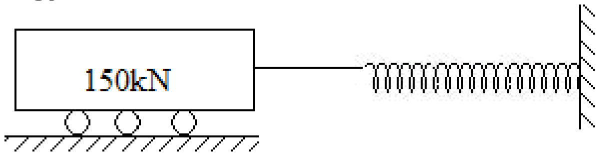
List-II
1. Percentage of elongation
2. Resistance to indentation
3. Ability to absorb energy during plastic deformation
4. Ability to be rolled into plates

Codes:	A	B	C	D		A	B	C	D
(a)	3	2	1	4	(b)	2	3	4	1
(c)	2	4	3	1	(d)	1	3	4	2

IAS-6. Ans. (b)

IAS-7. A truck weighing 150 kN and travelling at 2m/sec impacts which a buffer spring which compresses 1.25cm per 10 kN. The maximum compression of the spring is: **[IAS-1995]**

(a) 20.00 cm
(b) 22.85 cm
(c) 27.66 cm
(d) 30.00 cm



IAS-7. Ans. (c) Kinetic energy of the truck = strain energy of the spring

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2 \text{ or } x = \sqrt{\frac{mv^2}{k}} = \sqrt{\frac{\left(\frac{150\times 10^3}{9.81}\right)\times 2^2}{\left[\frac{10\times 1000}{0.0125}\right]}} = 0.2766\,\text{m} = 27.66\,\text{cm}$$

Previous Conventional Questions with Answers

Conventional Question IES 2009

Q. **A close coiled helical spring made of wire diameter d has mean coil radius R, number of turns n and modulus of rigidity G. The spring is subjected to an axial compression W.**

(1) Write the expression for the stiffness of the spring.

(2) What is the magnitude of the maximum shear stress induced in the spring wire neglecting the curvature effect? **[2 Marks]**

Ans. **(1)** Spring stiffness, $K = \frac{W}{X} = \frac{Gd^4}{8nD^3}$

(2) Maximum shear stress, $\tau = \frac{8WD}{\pi d^3}$

Conventional Question IES 2010

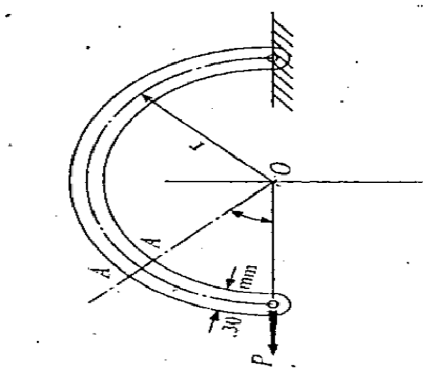
Q. **A semicircular steel ring of mean radius 300 mm is suspended vertically with the top end fixed as shown in the above figure and carries a vertical load of 200 N at the lowest point.**

Calculate the vertical deflection of the lower end if the ring is of rectangular cross- section 20 mm thick and 30 mm wide.

Value of Elastic modulus is $2\times10^5\ \text{N/mm}^2$**.**

Influence of circumferential and shearing forces may be neglected.

[10 Marks]



Ans. Load applied, F = 200 N

Mean Radius, R = 300 mm

Elastic modules, E = $2\times10^5\ \text{N/mm}^2$

I = Inertia of moment of cross – section

$$I = \frac{bd^3}{12} \qquad b = 20\text{ mm}$$

$$d = 30\text{ mm}$$

$$= \frac{20\times(30)^3}{12} = 45{,}000\text{ mm}^4$$

$\Rightarrow$ Influence of circumferential and shearing force are neglected strain energy at the section.

$$u = \int_0^{\pi} \frac{M^2 R d\theta}{2EI} \quad \underline{\text{for}} \ \frac{R}{4} \geq 10$$

$$M = F \times R \sin\theta$$

$$\Rightarrow \qquad \frac{\partial M}{\partial F} = R\ \sin\theta$$

$$\delta = \frac{\partial u}{\partial F} = \int_0^{\pi} \frac{FR^2 \sin^2\theta}{EI} d\theta \ \Rightarrow \ \frac{FR^2}{2EI} \times \pi$$

$$\delta = \frac{\pi FR^2}{2EI} = \frac{\pi \times 200 \times (300)^2}{2 \times 2 \times 10^5 \times 45000}$$

$$\delta = 3.14 \times 10^{-3} \text{ mm.}$$

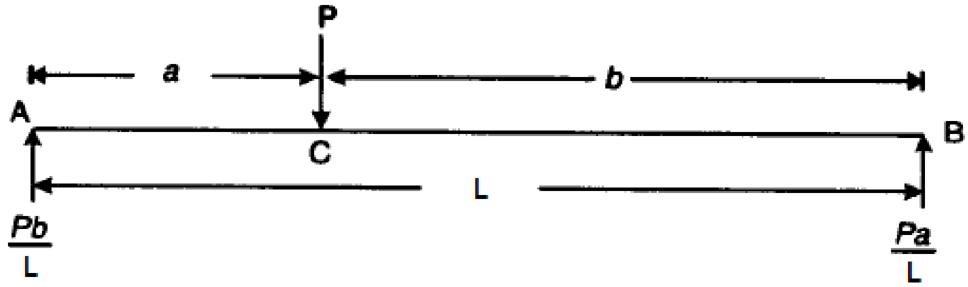
Conventional Question GATE-1996

Question: **A simply supported beam is subjected to a single force P at a distance b from one of the supports. Obtain the expression for the deflection under the load using Castigliano's theorem. How do you calculate deflection at the mid-point of the beam?**

Answer: Let load P acts at a distance b from the support B, and L be the total length of the beam.

Re*action at A*, $\quad R_A = \frac{Pb}{L}$, *and*

Re*action at A*, $\quad R_B = \frac{Pa}{L}$



Strain energy stored by beam AB,

U = Strain energy stored by AC (U_{AC}) + strain energy stored by BC (U_{BC})

$$= \int_0^a \left(\frac{Pb}{L}.x\right)^2 \frac{dx}{2EI} + \int_0^b \left(\frac{Pa}{L}.x\right)^2 \frac{dx}{2EI} = \frac{P^2b^2a^3}{6EIL^2} + \frac{P^2b^2a^3}{6EIL^2}$$

$$= \frac{P^2b^2a^2}{6EIL^2}(a+b) = \frac{P^2b^2a^2}{6EIL} = \frac{P^2(L-b)^2 b^2}{6EIL} \qquad [\because (a+b) = L)]$$

Deflection under the load P, $\delta = y = \frac{\partial U}{\partial P} = \frac{2P(L-b)^2 b^2}{6EIL} = \frac{P(L-b)^2 b^2}{3EIL}$

Deflection at the mid-span of the beam can be found by Macaulay's method.

By Macaulay's method, deflection at any section is given by

$$EIy = \frac{Pbx^3}{6L} - \frac{Pb}{6L}\left(L^2 - b^2\right)x - \frac{P\left(x-a\right)^3}{6}$$

Where y is deflection at any distance x from the support.

At $\quad x = \frac{L}{2}$, $i,e.$ at mid-span,

$$EIy = \frac{Pb \times \left(L/2\right)^3}{6L} - \frac{Pb}{6L}\left(L^2 - b^2\right) \times \frac{L}{2} - \frac{P\left(\frac{L}{2} - a\right)^3}{6}$$

or, $\quad EIy = \frac{PbL^2}{48} - \frac{Pb\left(L^2 - b^2\right)}{12} - \frac{P\left(L - 2a\right)^3}{48}$

$$y = \frac{P}{48EI}\left[bL^2 - 4b\left(L^2 - b^2\right) - \left(L - 2a\right)^3\right]$$

15. Theories of Failure

Theory at a Glance (for IES, GATE, PSU)

1. Introduction

- ***Failure:*** Every material has certain strength, expressed in terms of stress or strain, beyond which it fractures or fails to carry the load.
- **Failure Criterion:** A criterion used to hypothesize the failure.
- **Failure Theory:** A Theory behind a failure criterion.

Why Need Failure Theories?

- To design structural components and calculate margin of safety.
- To guide in materials development.
- To determine weak and strong directions.

Failure Mode

- **Yielding:** a process of global permanent plastic deformation. Change in the geometry of the object.
- **Low stiffness:** excessive elastic deflection.
- **Fracture:** a process in which cracks grow to the extent that the component breaks apart.
- **Buckling:** the loss of stable equilibrium. Compressive loading can lead to bucking in columns.
- **Creep:** a high-temperature effect. Load carrying capacity drops.

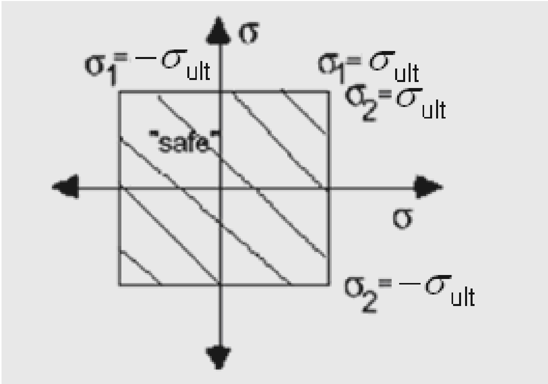
Failure Modes:

Excessive elastic deformation	**Yielding**	**Fracture**
1. Stretch, twist, or bending 2. Buckling 3. Vibration	• Plastic deformation at room temperature • Creep at elevated temperatures • Yield stress is the important design factor	• Sudden fracture of brittle materials • Fatigue (progressive fracture) • Stress rupture at elevated temperatures • Ultimate stress is the important design factor

2. Maximum Principal Stress Theory
(W. Rankin's Theory- 1850) – Brittle Material

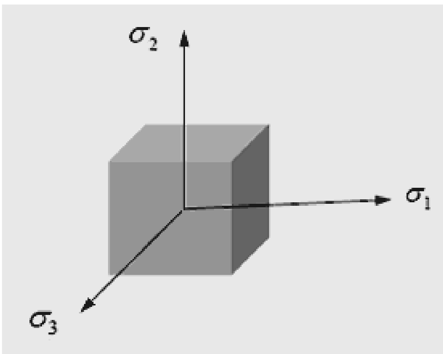
The maximum principal stress criterion:

- Rankin stated max principal stress theory as follows- a material fails by fracturing when the largest principal stress exceeds the ultimate strength σ_u in a simple tension test. That is, at the onset of fracture, $|\sigma_1| = \sigma_u$ OR $|\sigma_3| = \sigma_u$
- Crack will start at the most highly stressed point in a brittle material when the largest principal stress at that point reaches σ_u
- Criterion has good experimental verification, even though it assumes ultimate strength is same in compression and tension



Failure surface according to maximum principal stress theory

- This theory of yielding has very poor agreement with experiment. However, the theory has been used successfully for brittle materials.
- Used to describe fracture of **brittle materials** such as cast iron
- ***Limitations***
    - Doesn't distinguish between tension or compression
    - Doesn't depend on orientation of principal planes so only applicable to isotropic materials
- Generalization to 3-D stress case is easy:



3. Maximum Shear Stress or Stress difference theory (Guest's or Tresca's Theory-1868)- Ductile Material

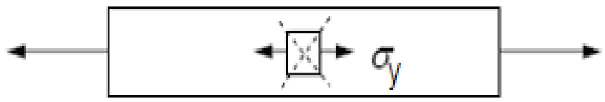
***The Tresca Criterion*:**

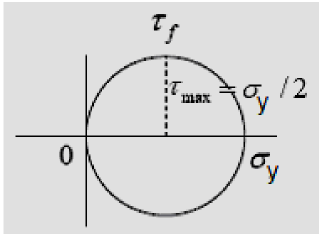
- Also known as the *Maximum Shear Stress* criterion.
- Yielding will occur when the maximum shear stress reaches that which caused yielding in a simple tension test.

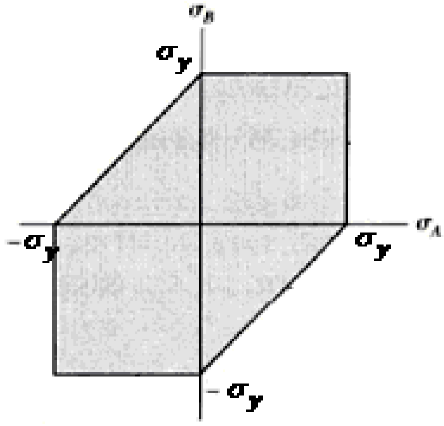
- Recall that yielding of a material occurred by slippage between planes oriented at 45° to principal stresses. This should indicate to you that yielding of a material depends on the maximum shear stress in the material rather than the maximum normal stress.

  If $\sigma_1 > \sigma_2 > \sigma_3$ Then $\sigma_1 - \sigma_3 = \sigma_y$

- Failure by slip (yielding) occurs when the maximum shearing stress, τ_{max} exceeds the yield stress τ_f as determined in a uniaxial tension test.
- This theory gives *satisfactory* result for ***ductile material***.







Failure surface according to maximum shear stress theory

4. Strain Energy Theory (Haigh's Theory)

The theory associated with Haigh

This theory is based on the assumption that strains are recoverable up to the elastic limit, and the energy absorbed by the material at failure up to this point is a single valued function independent of the stress system causing it. The strain energy per unit volume causing failure is equal to the strain energy at the elastic limit in simple tension.

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right] = \frac{\sigma_y^2}{2E}$$

$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) = \sigma_y^2$ For 3D- stress

$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \sigma_y^2$ For 2D- stress

5. Shear Strain Energy Theory (Distortion Energy Theory or Mises-Henky Theory or Von-Misses Theory)-Ductile Material

Von-Mises Criterion:

- Also known as the Maximum Energy of Distortion criterion
- Based on a more complex view of the role of the principal stress differences.

- In simple terms, the von Mises criterion considers the diameters of all three Mohr's circles as contributing to the characterization of yield onset in isotropic materials.
- When the criterion is applied, its relationship to the uniaxial tensile yield strength is:

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_y^2$$

- For a state of plane stress (σ_3 =0)

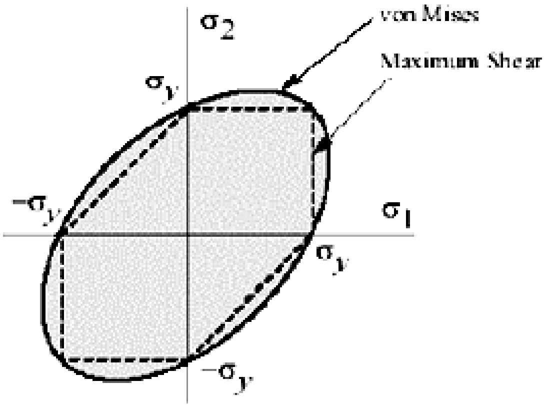
$$\sigma_1^{\,2}-\sigma_1\sigma_2+\sigma_2^{\,2}=\sigma_y^2$$

- It is often convenient to express this as an equivalent stress, σ_e:

$$\sigma_e=\frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}$$

$$or\,\sigma_e=\frac{1}{\sqrt{2}}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_x-\sigma_z)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)\right]^{1/2}$$

- In formulating this failure theory we used generalized Hooke's law for an isotropic material so the theory given is only applicable to those materials but it can be generalized to anisotropic materials.
- The von Mises theory is a little less conservative than the Tresca theory but in most cases there is little difference in their predictions of failure. Most experimental results tend to fall on or between these two theories.
- It gives very good result in **ductile material.**



6. Maximum Principal Strain Theory (St. Venant Theory)

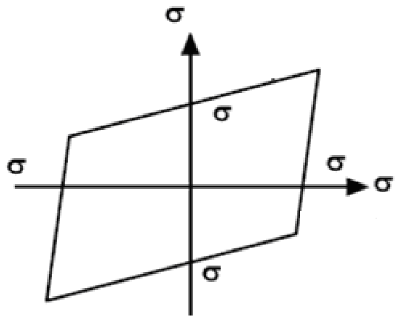
According to this theory, yielding will occur when the maximum principal strain just exceeds the strain at the tensile yield point in either simple tension or compression. If ε_1 and ε_2 are maximum and minimum principal strains corresponding to σ_1 and σ_2, in the limiting case

$$\varepsilon_1=\frac{1}{E}(\sigma_1-v\sigma_2) \qquad |\sigma_1|\geq|\sigma_2|$$

$$\varepsilon_2=\frac{1}{E}(\sigma_2-v\sigma_1) \qquad |\sigma_2|\geq|\sigma_1|$$

This gives, $E\varepsilon_1=\sigma_1-v\sigma_2=\pm\sigma_y$

$$E\varepsilon_2=\sigma_2-v\sigma_1=\pm\sigma_y$$

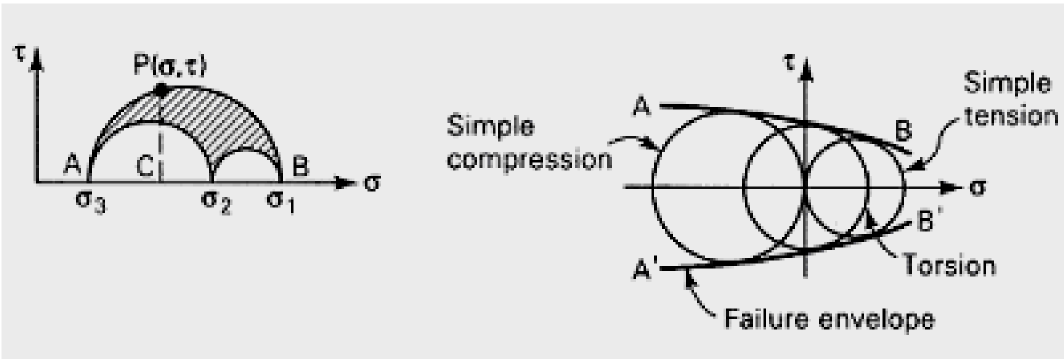


Yield surface corresponding to maximum principal strain theory

7. Mohr's theory- Brittle Material

Mohr's Theory

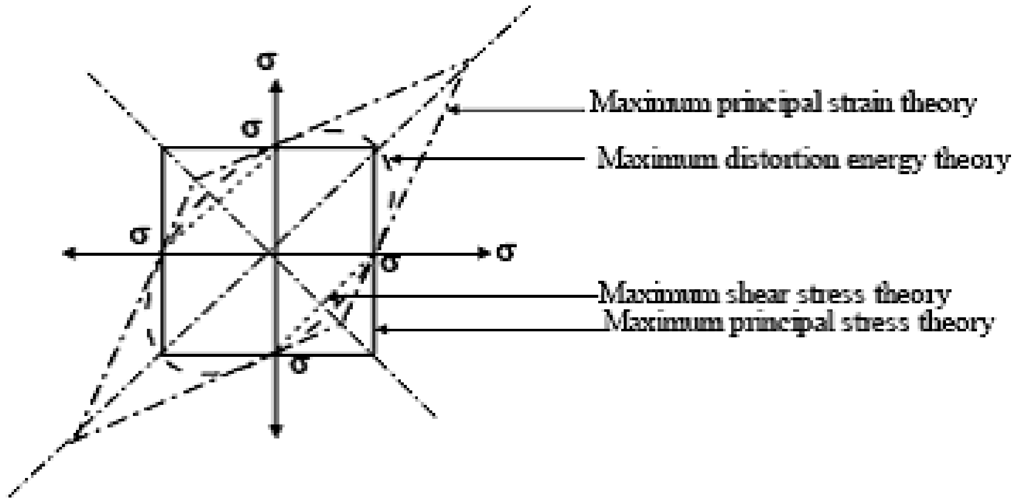
- Mohr's theory is used to predict the fracture of a material having different properties in tension and compression. Criterion makes use of Mohr's circle
- In Mohr's circle, we note that τ depends on σ, or $\tau = f(\sigma)$. Note the vertical line *PC* represents states of stress on planes with same σ but differing τ, which means the weakest plane is the one with maximum τ, point ***P***.
- Points on the outer circle are the weakest planes. On these planes the maximum and minimum principal stresses are sufficient to decide whether or not failure will occur.
- Experiments are done on a given material to determine the states of stress that result in failure. Each state defines a Mohr's circle. If the data are obtained from simple tension, simple compression, and pure shear, the three resulting circles are adequate to construct an ***envelope*** (AB & A'B')
- Mohr's envelope thus represents the locus of all possible failure states.



Higher shear stresses are to the left of origin, since most brittle materials have higher strength in compression

8. Comparison

A comparison among the different failure theories can be made by superposing the yield surfaces as shown in figure


σ
σ
σ
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σ
Maximum principal strain theory
Maximum distortion energy theory
Maximum shear stress theory
Maximum principal stress theory

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Maximum Shear stress or Stress Difference Theory

GATE-1. Match 4 correct pairs between list I and List II for the questions [GATE-1994]

List-I	List-II
(a) Hooke's law	**1.** Planetary motion
(b) St. Venant's law	**2.** Conservation Energy
(c) Kepler's laws	**3.** Elasticity
(d) Tresca's criterion	**4.** Plasticity
(e) Coulomb's laws	**5.** Fracture
(f) Griffith's law	**6.** Inertia

GATE-1. Ans. (a) - 3, (c) -1, (d) -5, (e) -2

St. Venant's law: Maximum principal strain theory

GATE-2. Which theory of failure will you use for aluminium components under steady loading? [GATE-1999]

(a) Principal stress theory (b) Principal strain theory
(c) Strain energy theory (d) Maximum shear stress theory

GATE-2. Ans. (d) Aluminium is a ductile material so use maximum shear stress theory

Shear Strain Energy Theory (Distortion energy theory)

GATE-3. According to Von-Mises' distortion energy theory, the distortion energy under three dimensional stress state is represented by [GATE-2006]

(a) $\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2v(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3\right]$

(b) $\frac{1-2v}{6E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

(c) $\frac{1+v}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

(d) $\frac{1}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-v(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

GATE-3. Ans. (c)

$$V_s=\frac{1}{12G}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}\quad \text{Where } E=2G(1+\mu) \text{ simplify and get result.}$$

GATE-4. A small element at the critical section of a component is in a bi-axial state of stress with the two principal stresses being 360 MPa and 140 MPa. The maximum working stress according to Distortion Energy Theory is: [GATE-1997]

(a) 220 MPa (b) 110 MPa (c) 314 MPa (d) 330 MPa

GATE-4. Ans. (c) According to distortion energy theory if maximum stress (σ_t) then

or $\sigma_t^2=\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2$

or $\sigma_t^2=360^2+140^2-360\times140$

or $\sigma_t=314$ MPa

Previous 20-Years IES Questions

Maximum Principal Stress Theory

IES-1. Match List-I (Theory of Failure) with List-II (Predicted Ratio of Shear Stress to Direct Stress at Yield Condition for Steel Specimen) and select the correct answer using the code given below the Lists: [IES-2006]

List-I	List-II
A. Maximum shear stress theory	**1.** 1·0
B. Maximum distortion energy theory	**2.** 0·577
C. Maximum principal stress theory	**3.** 0·62
D. Maximum principal strain theory	**4.** 0·50

Codes:	A	B	C	D		A	B	C	D
(a)	1	2	4	3	(b)	4	3	1	2
(c)	1	3	4	2	(d)	4	2	1	3

IES-1. Ans. (d)

IES-2. From a tension test, the yield strength of steel is found to be 200 N/mm². Using a factor of safety of 2 and applying maximum principal stress theory of failure, the permissible stress in the steel shaft subjected to torque will be: [IES-2000]
(a) 50 N/mm² (b) 57.7 N/mm² (c) 86.6. N/mm² (d) 100 N/mm²

IES-2. Ans. (d) For pure shear $\tau = \pm\sigma_x$

IES-3. A circular solid shaft is subjected to a bending moment of 400 kNm and a twisting moment of 300 kNm. On the basis of the maximum principal stress theory, the direct stress is σ and according to the maximum shear stress theory, the shear stress is τ. The ratio σ/ τ is: [IES-2000]

(a) $\frac{1}{5}$ (b) $\frac{3}{9}$ (c) $\frac{9}{5}$ (d) $\frac{11}{6}$

IES-3. Ans. (c) $\sigma = \frac{16}{\pi d^3}\left(M + \sqrt{M^2 + T^2}\right)$ and $\tau = \frac{16}{\pi d^3}\left(\sqrt{M^2 + T^2}\right)$

$$\text{Therefore} \frac{\sigma}{\tau} = \frac{M + \sqrt{M^2 + T^2}}{\sqrt{M^2 + T^2}} = \frac{4 + \sqrt{4^2 + 3^2}}{\sqrt{4^2 + 3^2}} = \frac{9}{5}$$

IES-4. A transmission shaft subjected to bending loads must be designed on the basis of [IES-1996]
(a) Maximum normal stress theory
(b) Maximum shear stress theory
(c) Maximum normal stress and maximum shear stress theories
(d) Fatigue strength

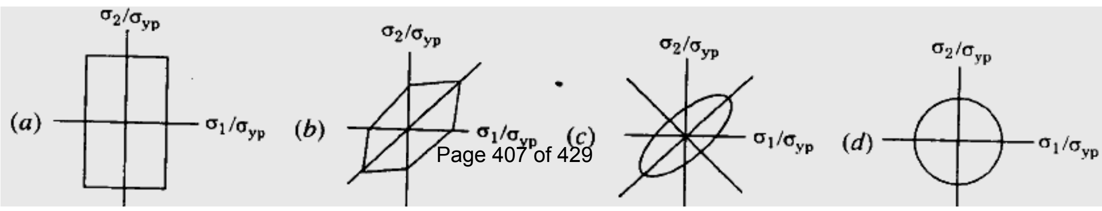
IES-4. Ans. (a)

IES-5. Design of shafts made of brittle materials is based on [IES-1993]
(a) Guest's theory (b) Rankine's theory (c) St. Venant's theory (d) Von Mises theory

IES-5. Ans. (b) Rankine's theory or maximum principle stress theory is most commonly used for brittle materials.

Maximum Shear stress or Stress Difference Theory

IES-6. Which one of the following figures represents the maximum shear stress theory or Tresca criterion? [IES-1999]



IES-6. Ans. (b)

IES-7. According to the maximum shear stress theory of failure, permissible twisting moment in a circular shaft is 'T'. The permissible twisting moment will the same shaft as per the maximum principal stress theory of failure will be: [IES-1998: ISRO-2008]

(a) T/2 (b) T (c) $\sqrt{2T}$ (d) 2T

IES-7. Ans. (d) Given $\tau = \frac{16T}{\pi d^3} = \frac{\sigma_{yt}}{2}$ principal stresses for only this shear stress are

$\sigma_{1,2} = \sqrt{\tau^2} = \pm\tau$ maximum principal stress theory of failure gives

$$\max[\sigma_1, \sigma_2] = \sigma_{yt} = \frac{16(2T)}{\pi d^3}$$

IES-8. Permissible bending moment in a circular shaft under pure bending is M according to maximum principal stress theory of failure. According to maximum shear stress theory of failure, the permissible bending moment in the same shaft is: [IES-1995]

(a) 1/2 M (b) M (c) $\sqrt{2}$ M (d) 2M

IES-8. Ans. (b) $\sigma = \frac{16}{\pi d^3}\left(M + \sqrt{M^2 + T^2}\right)$ and $\tau = \frac{16}{\pi d^3}\left(\sqrt{M^2 + T^2}\right)$ put T = 0

$$\text{or } \sigma_{yt} = \frac{32M}{\pi d^3} \text{ and } \tau = \frac{16M'}{\pi d^3} = \frac{\sigma_{yt}}{2} = \frac{\left(\frac{32M}{\pi d^3}\right)}{2} = \frac{16M}{\pi d^3} \text{ Therefore } M' = M$$

IES-9. A rod having cross-sectional area 100 x 10^{-6} m² is subjected to a tensile load. Based on the Tresca failure criterion, if the uniaxial yield stress of the material is 200 MPa, the failure load is: [IES-2001]

(a) 10 kN (b) 20 kN (c) 100 kN (d) 200 kN

IES-9. Ans. (b) Tresca failure criterion is maximum shear stress theory.

$$\text{We know that, } \tau = \frac{P}{A}\frac{\sin 2\theta}{2} \text{ or } \tau_{max} = \frac{P}{2A} = \frac{\sigma_{yt}}{2} \text{ or } P = \sigma_{yt} \times A$$

IES-10. A cold roller steel shaft is designed on the basis of maximum shear stress theory. The principal stresses induced at its critical section are 60 MPa and - 60 MPa respectively. If the yield stress for the shaft material is 360 MPa, the factor of safety of the design is: [IES-2002]

(a) 2 (b) 3 (c) 4 (d) 6

IES-10. Ans. (b)

IES-11. A shaft is subjected to a maximum bending stress of 80 N/mm² and maximum shearing stress equal to 30 N/mm² at a particular section. If the yield point in tension of the material is 280 N/mm², and the maximum shear stress theory of failure is used, then the factor of safety obtained will be: [IES-1994]

(a) 2.5 (b) 2.8 (c) 3.0 (d) 3.5

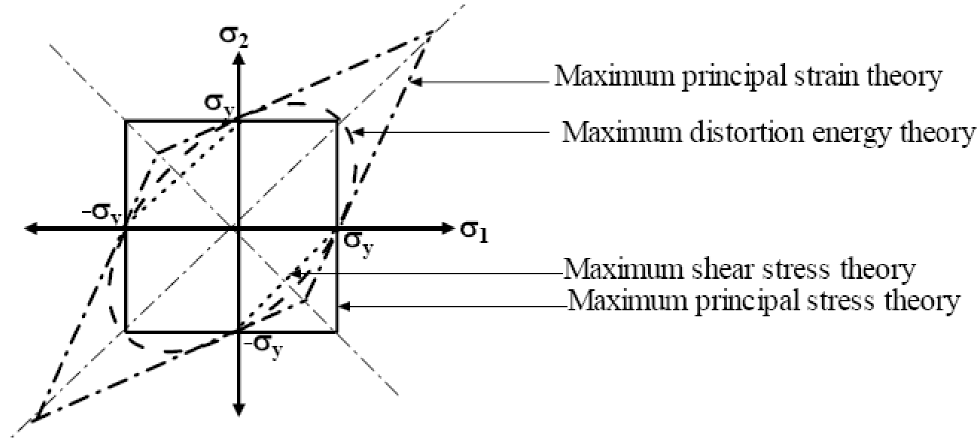
IES-11. Ans. (b) Maximum shear stress = $\sqrt{\left(\frac{80-0}{2}\right)^2 + 30^2} = 50\,\text{N/mm}^2$

According to maximum shear stress theory, $\tau = \frac{\sigma_y}{2};\ \therefore F.S. = \frac{280}{2\times 50} = 2.8$

IES-12. For a two-dimensional state stress ($\sigma_1 > \sigma_2, \sigma_1 > 0, \sigma_2 < 0$) the designed values are most conservative if which one of the following failure theories were used? [IES-1998]

(a) Maximum principal strain theory (b) Maximum distortion energy theory
(c) Maximum shear stress theory (d) Maximum principal stress theory

IES-12. Ans. (c)



Graphical comparison of different failure theories

Above diagram shows that $\sigma_1 > 0, \sigma_2 < 0$ will occur at 4th quadrant and most conservative design will be maximum shear stress theory.

Shear Strain Energy Theory (Distortion energy theory)

IES-13. Who postulated the maximum distortion energy theory? [IES-2008]

(a) Tresca (b) Rankine (c) St. Venant (d) Mises-Henky

IES-13. Ans. (d)

IES-14. Who postulated the maximum distortion energy theory? [IES-2008]

(a) Tresca (b) Rankine (c) St. Venant (d) Mises-Henky

IES-14. Ans. (d)

Maximum shear stress theory	$\rightarrow$	Tresca
Maximum principal stress theory	$\rightarrow$	Rankine
Maximum principal strain theory	$\rightarrow$	St. Venant
Maximum shear strain energy theory	$\rightarrow$	Mises – Henky

IES-15. The maximum distortion energy theory of failure is suitable to predict the failure of which one of the following types of materials? [IES-2004]

(a) Brittle materials (b) Ductile materials (c) Plastics (d) Composite materials

IES-15. Ans. (b)

IES-16. If σ_y is the yield strength of a particular material, then the distortion energy theory is expressed as [IES-1994]

(a) $(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_y^2$

(b) $(\sigma_1^2-\sigma_2^2+\sigma_3^2)-2\mu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)=\sigma_y^2$

(c) $(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=3\sigma_y^2$

(d) $(1-2\mu)(\sigma_1+\sigma_2+\sigma_3)^2=2(1+\mu)\sigma_y^2$

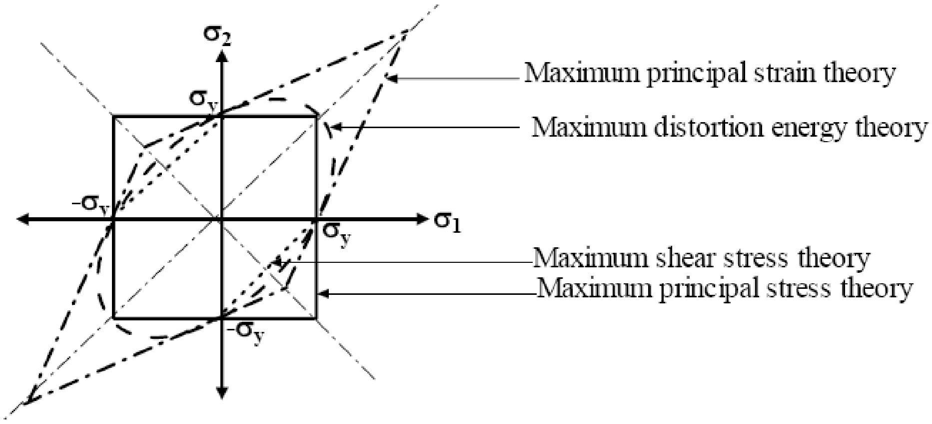
IES-16. Ans. (a)

IES-17. If a shaft made from ductile material is subjected to combined bending and twisting moments, calculations based on which one of the following failure theories would give the most conservative value? [IES-1996]

(a) Maximum principal stress theory (b) Maximum shear stress theory.

(d Maximum strain energy theory (d) Maximum distortion energy theory.

IES-17. Ans. (b)

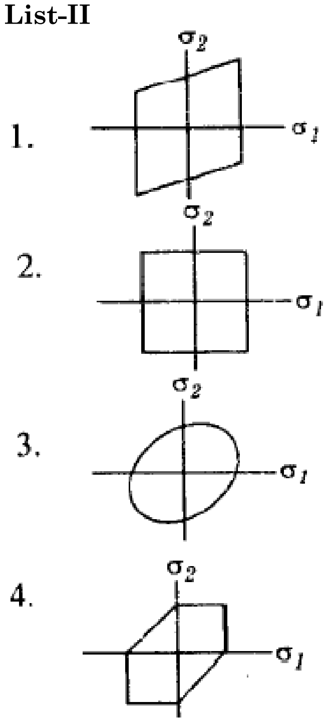


Maximum Principal Strain Theory

IES-18. Match List-I (Failure theories) with List-II (Figures representing boundaries of these theories) and select the correct answer using the codes given below the Lists: [IES-1997]

List-I
- **A.** Maximum principal stress theory
- **B.** Maximum shear stress theory
- **C.** Maximum octahedral stress theory
- **D.** Maximum shear strain energy theory

List-II



Code:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	2	1	3	4	(b)	2	4	3	1
(c)	4	2	3	1	(d)	2	4	1	3

IES-18. Ans. (d)

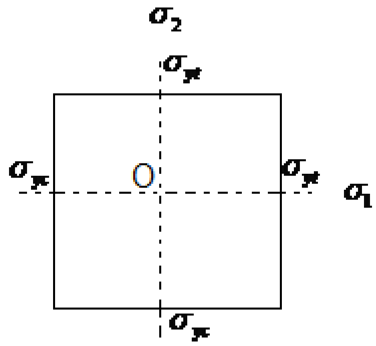
Previous 20-Years IAS Questions

Maximum Principal Stress Theory

IAS-1. For $\sigma_1 \neq \sigma_2$ and $\sigma_3 = 0$, what is the physical boundary for Rankine failure theory? [IAS-2004]

(a) A rectangle (b) An ellipse (c) A square (d) A parabola

IAS-1. Ans. (c) Rankine failure theory or Maximum principle stress theory.



Shear Strain Energy Theory (Distortion energy theory)

IAS-2. Consider the following statements: [IAS-2007]

1. Experiments have shown that the distortion-energy theory gives an accurate prediction about failure of a ductile component than any other theory of failure.

2. According to the distortion-energy theory, the yield strength in shear is less than the yield strength in tension.

Which of the statements given above is/are correct?

(a) 1 only (b) 2 only (c) Both 1 and 2 (d) Neither 1 nor 2

IAS-2. Ans. (c) $\tau_y = \dfrac{\sigma_y}{\sqrt{3}} = 0.577\sigma_y$

IAS-3. Consider the following statements: [IAS-2003]

1. Distortion-energy theory is in better agreement for predicting the failure of ductile materials.

2. Maximum normal stress theory gives good prediction for the failure of brittle materials.

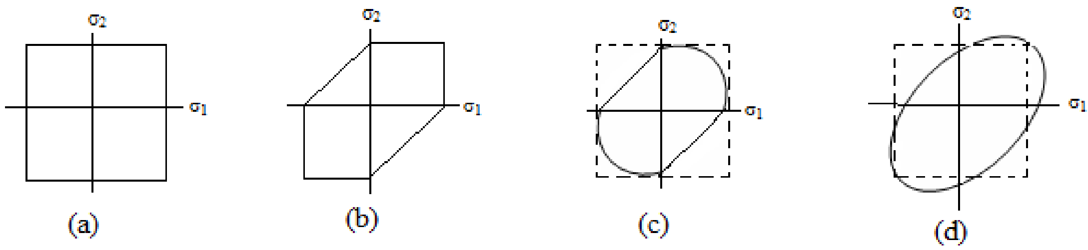
3. Module of elasticity in tension and compression are assumed to be different stress analysis of curved beams.

Which of these statements is/are correct?

(a) 1, 2 and 3 (b) 1 and 2 (c) 3 only (d) 1 and 3

IAS-3. Ans. (b)

IAS-4. Which one of the following graphs represents Mises yield criterion? [IAS-1996]



IAS-4. Ans. (d)

Maximum Principal Strain Theory

IAS-5. Given that the principal stresses $\sigma_1 > \sigma_2 > \sigma_3$ and σ_e is the elastic limit stress in simple tension; which one of the following must be satisfied such that the elastic failure does not occur in accordance with the maximum principal strain theory? [IAS-2004]

(a) $\dfrac{\sigma_e}{E} < \left(\dfrac{\sigma_1}{E} - \mu\dfrac{\sigma_2}{E} - \mu\dfrac{\sigma_3}{E}\right)$ (b) $\dfrac{\sigma_e}{E} > \left(\dfrac{\sigma_1}{E} - \mu\dfrac{\sigma_2}{E} - \mu\dfrac{\sigma_3}{E}\right)$

(c) $\dfrac{\sigma_e}{E} > \left(\dfrac{\sigma_1}{E} + \mu\dfrac{\sigma_2}{E} + \mu\dfrac{\sigma_3}{E}\right)$ (d) $\dfrac{\sigma_e}{E} < \left(\dfrac{\sigma_1}{E} + \mu\dfrac{\sigma_2}{E} - \mu\dfrac{\sigma_3}{E}\right)$

IAS-5. Ans. (b) Strain at yield point>principal strain

$$\frac{\sigma_e}{E} > \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$

Previous Conventional Questions with Answers

Conventional Question ESE-2010

Q. **The stress state at a point in a body is plane with**

$$\boldsymbol{\sigma_1 = 60\,N/mm^2 \;\&\; \sigma_2 = -36\,N/mm^2}$$

If the allowable stress for the material in simple tension or compression is 100 N/mm² calculate the value of factor of safety with each of the following criteria for failure

(i) Max Stress Criteria
(ii) Max Shear Stress Criteria
(iii) Max strain criteria
(iv) Max Distortion energy criteria **[10 Marks]**

Ans. The stress at a point in a body is plane

$$\sigma_1 = 60\ \ N/mm^2 \qquad \sigma_2 = -36\ \ N/mm^2$$

Allowable stress for the material in simple tension or compression is 100 N/mm²
Find out factor of safety for

(i) **Maximum stress Criteria : -** In this failure point occurs when max principal stress reaches the limiting strength of material.

Therefore. Let F.S factor of safety

$$\sigma_1 = \frac{\sigma\ (\text{allowable})}{F.S}$$

$$F.S = \frac{100\ N/mm^2}{60\ N/mm^2} = \underline{1.67} \qquad \text{Ans.}$$

(ii) **Maximum Shear stress criteria : -** According to this failure point occurs at a point in a member when maximum shear stress reaches to shear at yield point

$$\gamma_{max} = \frac{\sigma_{yt}}{2\ F.S} \qquad \sigma_{yt} = 100\ N/mm^2$$

$$\gamma_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{60+36}{2} = \frac{96}{2} = \underline{48}\ N/mm^2$$

$$48 = \frac{100}{2 \times F.S}$$

$$F.S = \frac{100}{\underline{2 \times 48}} = \frac{100}{96} = \underline{1.042}$$

$$F.S = 1.042 \qquad \text{Ans.}$$

(iii) **Maximum Strain Criteria !** – In this failure point occurs at a point in a member when maximum strain in a bi – axial stress system reaches the limiting value of strain (i.e strain at yield point)

$$\boldsymbol{\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \left(\frac{\sigma_{\text{allowable}}}{FOS}\right)^2}$$

$$\mathbf{FOS = 1.27}$$

$$\mathbf{(\mu = 0.3\,assume)}$$

(iv) **Maximum Distortion energy criteria !** – In this failure point occurs at a point in a member when distortion strain energy per unit volume in a bi – axial system reaches the limiting distortion strain energy at the of yield

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \times \sigma_2 = \left(\frac{\sigma_{yt}}{F.S}\right)^2$$

$$60^2 + (36)^2 - \times 60 \times -36 = \left(\frac{100}{F.S}\right)^2$$

$$F.S = 1.19$$

Conventional Question ESE-2006

Question: **A mild steel shaft of 50 mm diameter is subjected to a beading moment of 1.5 kNm and torque T. If the yield point of steel in tension is 210 MPa, find the maximum value of the torque without causing yielding of the shaft material according to**

(i) Maximum principal stress theory

(ii) Maximum shear stress theory.

Answer: We know that, Maximum bending stress $(\sigma_b) = \dfrac{32M}{\pi d^3}$

and Maximum shear stress $(\tau) = \dfrac{16T}{\pi d^3}$

Principal stresses are given by:

$$\sigma_{1,2} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} = \frac{16}{\pi d^3}\left[M \pm \sqrt{M^2 + T^2}\right]$$

(*i*) According to Maximum principal stress theory

Maximum principal stress=Maximum stress at elastic limit (σ_y)

$$or\ \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right] = 210 \times 10^6$$

$$\text{or}\ \frac{16}{\pi (0.050)^3}\left[1500 + \sqrt{1500^2 + T^2}\right] = 210 \times 10^6$$

or T = 3332 Nm = 3.332 kNm

(*ii*) According to Maximum shear stress theory

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_y}{2}$$

$$or,\ \sigma_1 - \sigma_2 = \sigma_y$$

$$or,\ 2 \times \frac{16}{\pi d^3}\sqrt{M^2 + T^2} = 210 \times 10^6$$

or, T = 2096 N m = 2.096 kNm

Conventional Question ESE-2005

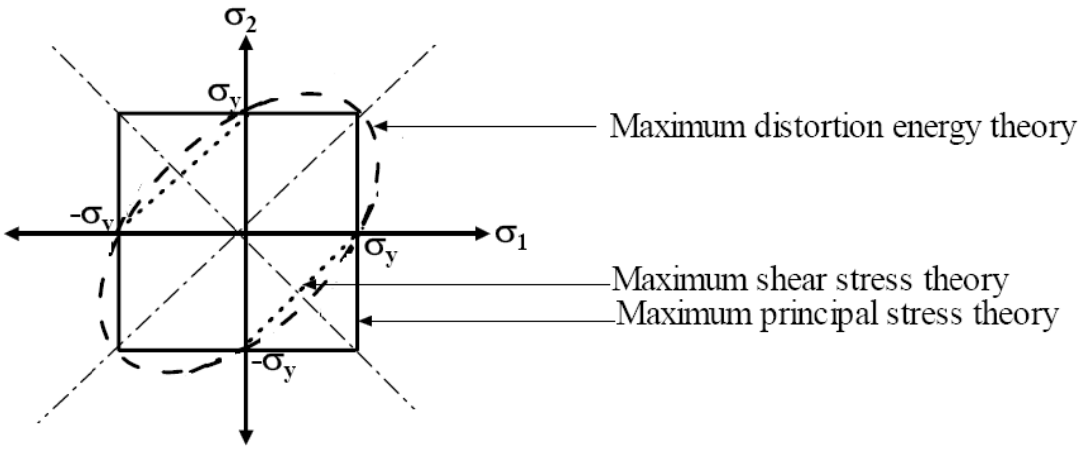
Question: **Illustrate the graphical comparison of following theories of failures for two-dimensional stress system:**

(i) Maximum normal stress theory

(ii) Maximum shear stress theory

(iii) Distortion energy theory

Answer:



Conventional Question ESE-2004

Question: **State the Von- Mises's theory. Also give the naturally expression.**

Answer: According to this theory yielding would occur when total distortion energy absorbed per unit volume due to applied loads exceeds the distortion energy absorbed per unit volume at the tensile yield point. The failure criterion is

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_y^2$$

[symbols has usual meaning]

Conventional Question ESE-2002

Question: **Derive an expression for the distortion energy per unit volume for a body subjected to a uniform stress state, given by the σ_1 and σ_2 with the third principal stress σ_3 being zero.**

Answer: According to this theory yielding would occur when total distortion energy absorbed per unit volume due to applied loads exceeds the distortion energy absorbed per unit volume at the tensile yield point. Total strain energy E_T and strain energy for volume change E_V can be given as

$$E_T=\frac{1}{2}(\sigma_1\varepsilon_1+\sigma_2\varepsilon_2+\sigma_3\varepsilon_3) \text{ and } E_V=\frac{3}{2}\sigma_{av}\varepsilon_{av}$$

Substituting strains in terms of stresses the distortion energy can be given as

$$E_d = E_T - E_V = \frac{2(1+\nu)}{6E}\left(\sigma_1{}^2+\sigma_2{}^2+\sigma_3{}^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1\right)$$

At the tensile yield point, $\sigma_1 = \sigma_y$, $\sigma_2 = \sigma_3 = 0$ which gives

$$E_{dy}=\frac{2(1+\nu)}{6E}\sigma_y{}^2$$

The failure criterion is thus obtained by equating E_d and E_{dy} , which gives

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_y^2$$

In a 2-D situation if $\sigma_3 = 0$, the criterion reduces to

$$\sigma_1{}^2+\sigma_2{}^2-\sigma_1\sigma_2=\sigma_y{}^2$$

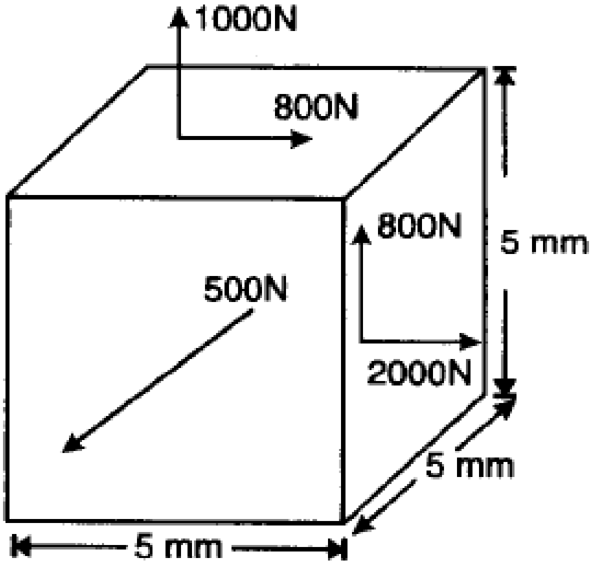
Conventional Question GATE-1996

Question: **A cube of 5mm side is loaded as shown in figure below.**

(i) Determine the principal stresses $\sigma_1, \sigma_2, \sigma_3$.

(ii) Will the cube yield if the yield strength of the material is 70 MPa? Use Von-Mises theory.

Answer: Yield strength of the material σ_{et} = 70 MPa = 70 MN/m² or 70 N/mm².



(i) Principal stress $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_x = \frac{2000}{5\times 5} = 80 \text{ N/mm}^2; \qquad \sigma_y = \frac{1000}{5\times 5} = 40 \text{ N/mm}^2$$

$$\sigma_z = \frac{500}{5\times 5} = 20 \text{ N/mm}^2; \qquad \tau_{xy} = \frac{800}{5\times 5} = 32 \text{ N/mm}^2$$

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right) + \tau_{xy}^2} = \frac{80+40}{2} \pm \sqrt{\left(\frac{80-40}{2}\right)^2 + (32)^2}$$

$$= 60 \pm \sqrt{(20)^2 + (32)^2} = 97.74, 22.26$$

$\therefore \quad \sigma_1 = 97.74 \text{ N/mm}^2$, or 97.74 MPa

and $\quad \sigma_2 = 22.96 \text{ N/mm}^2$ or 22.96 MPa

$\sigma_3 = \sigma_z = 20 \text{ N/mm}^2$ or 22 MPa

(ii) Will the cube yield or not?

According to Von-Mises yield criteria, yielding will occur if

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2\sigma_{yt}^2$$

Now $\quad (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$

$$= (97.74 - 22.96)^2 + (22.96 - 20)^2 + (20 - 97.74)^2$$

$$= 11745.8 \qquad ---(i)$$

and, $\quad 2\sigma_{yt}^2 = 2 \times (70)^2 = 9800 \qquad ---(ii)$

Since 11745.8 > 9800 so yielding will occur.

Conventional Question GATE-1995

***Question*:** **A thin-walled circular tube of wall thickness t and mean radius r is subjected to an axial load P and torque T in a combined tension-torsion experiment.**
(i) Determine the state of stress existing in the tube in terms of P and T.
(ii) Using Von-Mises - Henky failure criteria show that failure takes place when $\sqrt{\sigma^2 + 3\tau^2} = \sigma_0$, *where* σ_0 *is the yield stress in uniaxial tension,* σ *and* τ *are respectively the axial and torsional stresses in the tube.*

***Answer*:** Mean radius of the tube = r,
Wall thickness of the tube = t,
Axial load = P, and
Torque = T.
(i) The state of stress in the tube:

Due to axial load, the axial stress in the tube $\sigma x = \dfrac{P}{2\pi rt}$

Due to torque, shear stress,

$$\tau_{xy} = \frac{Tr}{J} = \frac{Tr}{2\pi r^3 t} = \frac{T}{2\pi r^3 t}$$

$$J = \frac{\pi}{2}\left\{(r+t)^4 - r^4\right\} = 2\pi r^3 t \text{-neglecting } t^2 \text{ higher power of t.}$$

$\therefore$The state of stress in the tube is, $\sigma_x = \dfrac{P}{2\pi rt}, \sigma_y = 0, \tau_{xy} = \dfrac{T}{2\pi r^3 t}$

(ii) Von Mises-Henky failure in tension for 2-dimensional stress is

$$\sigma_0^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

In this case, $\sigma_1 = \dfrac{\sigma_x}{2} + \sqrt{\dfrac{\sigma_x^2}{4} + \tau_{xy}^2}$, and

$$\sigma_2 = \frac{\sigma_x}{2} - \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2} \qquad (\because \sigma_y = 0)$$

$$\therefore \sigma_0^2 = \left[\frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]^2 + \left[\frac{\sigma_x}{2} - \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]^2 - \left[\frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]\left[\frac{\sigma_x}{2} - \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]^2$$

$$= \left[\frac{\sigma_x^2}{4} + \frac{\sigma_x^2}{4} + \tau_{xy}^2 + 2.\frac{\sigma_x}{2}.\sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right] + \left[\frac{\sigma_x^2}{4} + \frac{\sigma_x^2}{4} + \tau_{xy}^2 + 2.\frac{\sigma_x}{2}.\sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]$$

$$-\left[\frac{\sigma_x^2}{4} - \frac{\sigma_x^2}{4} - \tau_{xy}^2\right]$$

$$= \sigma_x^2 + 3\tau_{xy}^2$$

$$\sigma_0 = \sqrt{\sigma_x^2 + 3\tau_{xy}^2}$$

Conventional Question GATE-1994

Question: **Find the maximum principal stress developed in a cylindrical shaft. 8 cm in diameter and subjected to a bending moment of 2.5 kNm and a twisting moment of 4.2 kNm. If the yield stress of the shaft material is 300 MPa. Determine the factor of safety of the shaft according to the maximum shearing stress theory of failure.**

Answer: Given: d = 8 cm = 0.08 m; M = 2.5 kNm = 2500 Nm; T = 4.2 kNm = 4200 Nm

$$\sigma_{yield}\left(\sigma_{yt}\right) = 300\ MPa = 300 \text{ MN/m}^2$$

Equivalent torque, $T_e = \sqrt{M^2 + T^2} = \sqrt{(2.5)^2 + (4.2)^2} = 4.888 \text{ kNm}$

Maximum shear stress developed in the shaft,

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16 \times 4.888 \times 10^3}{\pi \times (0.08)^3} \times 10^{-6} \text{ MN/m}^2 = 48.62 \text{MN/m}^2$$

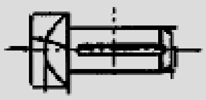
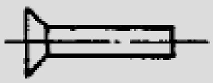
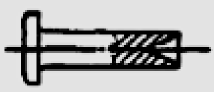
$$\text{Permissible shear stress} = \frac{300}{2} = 150 \text{MN/m}^2$$

$$\therefore \quad \text{Factor of safety} = \frac{150}{48.62} = 3.085$$

16. Riveted and Welded Joint

Theory at a Glance (for IES, GATE, PSU)

Types of Rivets

FIGURE	DESCRIPTION	
		LARGE BUTTON HEAD
		NARROW BUTTON HEAD
		NARROW BUTTON HEAD
		BUTTON HEAD WITH SPLINED SHANK
		LARGE BUTTON HEAD
		COUNTERSUNK HEAD
		FLAT HEAD
	RIVETS	THIN NARROW BUTTON HEAD, SEMITUBULAR
		COUNTERSUNK FLAT HEAD, SEMITUBULAR
		FLAT HEAD, SEMITUBULAR

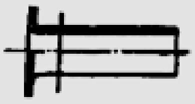
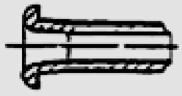
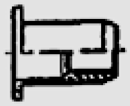
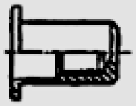
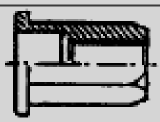
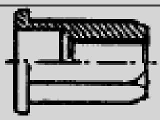
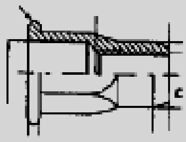
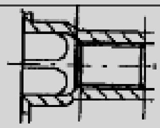
		FLAT HEAD SEMITUBULAR, SHOULDER
		FLAT HEAD, TUBULAR
		FLAT HEAD, TUBULAR
		FLAT HEAD THREADED, TUBULAR
		FLAT HEAD, CLOSED END, THREADED TUBULAR HEXAGON

FIGURE	DESCRIPTION	
	RIVETS	HEXAGON THREADED FLAT HEAD
		HEXAGON CLOSE END THREADED FLAT HEAD
		THREADED HEXAGON FOR AUTOMATED ASSEMBLY
		SQUARE THREADED FLAT HEAD
		BUTTON HEAD, TAPPING

OBJECTIVE QUESTIONS (GATE, IES, IAS)

Previous 20-Years GATE Questions

Failure of riveted joint

GATE-1. Bolts in the flanged end of pressure vessel are usually pre-tensioned Indicate which of the following statements is NOT TRUE? [GATE-1999]

(a) Pre-tensioning helps to seal the pressure vessel

(b) Pre-tensioning increases the fatigue life of the bolts

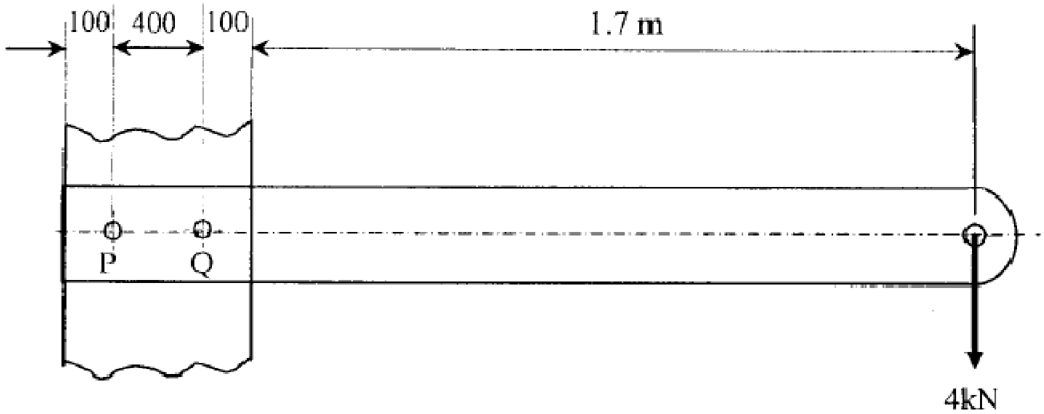
(c) Pre-tensioning reduces the maximum tensile stress in the bolts

(d) Pre-tensioning helps to reduce the effect of pressure pulsations in the pressure vessel

GATE-1. Ans. (c)

Statement for Linked Answers and Questions Q2 and Q3

A steel bar of 10 × 50 mm is cantilevered with two M 12 bolts (P and Q) to support a static load of 4 kN as shown in the figure.



GATE-2. The primary and secondary shear loads on bolt P, respectively, are: [GATE-2008]

(A) 2 kN, 20 kN (B) 20 kN, 2kN (C) 20kN,0kN (D) 0kN, 20 kN

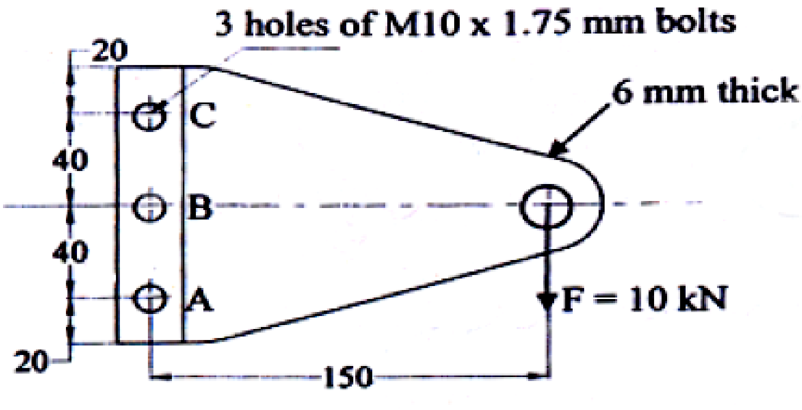
GATE-2. Ans. (a) Primary (Direct) Shear load = $\frac{4\,kN}{2} = 2\,k$ N

GATE-3. The resultant stress on bolt P is closest to [GATE-2008]

(A) 132 MPa (B) 159 MPa (C) 178 MPa (D) 195 MPa

GATE-3. Ans. (b)

GATE-4. A bolted joint is shown below. The maximum shear stress, in MPa, in the bolts at A and B, respectively are: [GATE-2007]

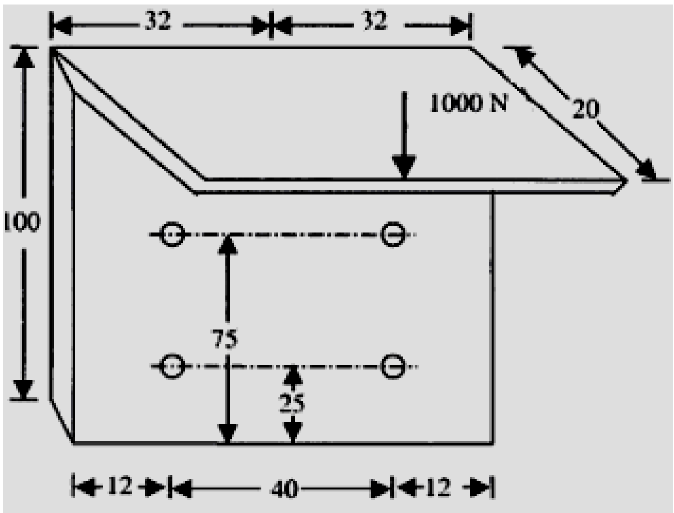


(all dimensions in the figure are in mm)

(a) 242.6, 42.5 (b) 425.5, 242.6 (c) 42.5, 42.5 (d) 242.6, 242.6

GATE-4. Ans. (a)

GATE-5. A bracket (shown in figure) is rigidly mounted on wall using four rivets. Each rivet is 6 mm in diameter and has an effective length of 12 mm. [GATE-2010]



Direct shear stress (in MPa) in the most heavily loaded rivet is:

(a) 4.4 (b) 8.8 (c) 17.6 (d) 35.2

GATE-5. Ans. (b)

$$F = \frac{1000}{4} = 250\,\text{N} \quad \text{and} \quad z = \frac{F}{A} = \frac{250}{\frac{\pi}{4}(6)^2} = 8.8\,\text{MPa}$$

Efficiency of a riveted joint

GATE-6. If the ratio of the diameter of rivet hole to the pitch of rivets is 0.25, then the tearing efficiency of the joint is: [GATE-1996]

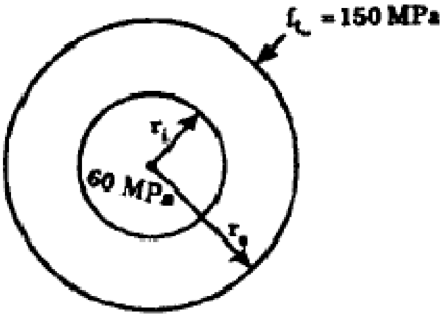
(a) 0.50 (b) 0.75 (c) 0.25 (d) 0.87

GATE-6. Ans. (b)

$$\frac{d}{P} = 0.25$$

$$\eta_{tearing} = \left(\frac{P-d}{P}\right) \times 100$$

$$= 75\%$$



GATE-7. A manufacturer of rivets claims that the failure load in shear of his product is 500 ± 25 N. This specification implies that [GATE-1992]

(a) No rivet is weaker than 475 N and stronger than 525 N
(b) The standard deviation of strength of random sample of rivets is 25 N
(c) There is an equal probability of failure strength to be either 475 Nor 525 N
(d) There is approximately two-to-one chance that the strength of a rivet lies between 475 N to 525 N

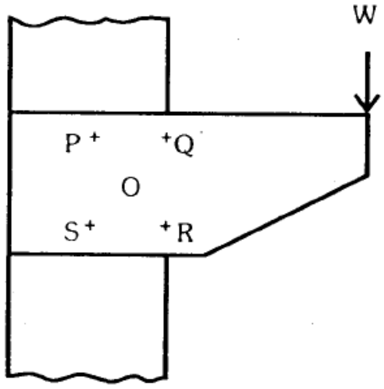
GATE-7. Ans. (a)

Previous 20-Years IES Questions

Failure of riveted joint

IES-1. An eccentrically loaded riveted joint is shown with 4 rivets at P, Q, R and S. Which of the rivets are the most loaded?

(a) P and Q
(b) Q and R
(c) Rand S
(d) Sand P



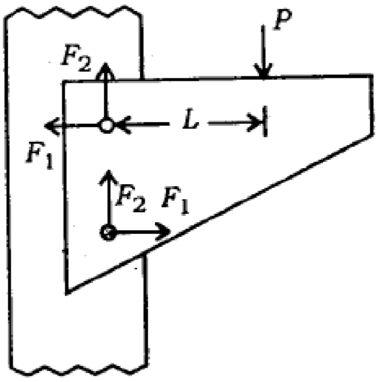
[IES-2002]

IES-1. Ans. (b)

IES-2. A riveted joint has been designed to support an eccentric load P. The load generates value of F_1 equal to 4 kN and F_2 equal to 3 kN. The cross-sectional area of each rivet is 500 mm². Consider the following statements:

1. The stress in the rivet is 10 N / mm²
2. The value of eccentricity L is 100 mm
3. The value of load P is 6 kN
4. The resultant force in each rivet is 6 kN
Which of these statements are correct?

(a) 1 and 2 (b) 2 and 3
(c) 3 and 4 (d) 1 and 3



[IES-2003]

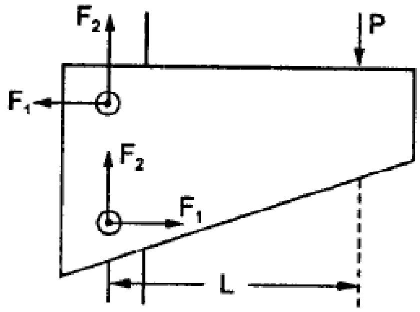
IES-2. Ans. (d)

$P = 2F_2 = 2 \times 3 = 6$ kN

and $P.L = F_1 l + F_1 l = 2F_1 l$

or $6L = 2 \times 4l = 8l$

or $\frac{L}{l} = \frac{8}{6}$

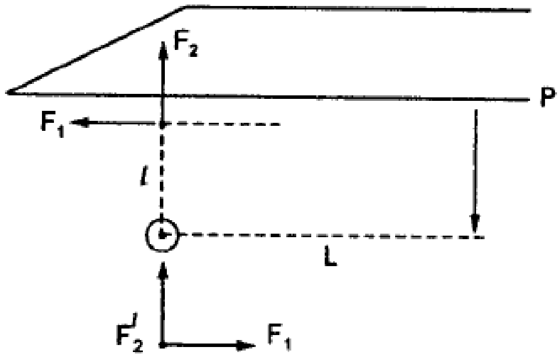


Resultant force on rivet,

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

$$= \sqrt{(4)^2 + (3)^2 + 2 \times 4 \times 3\cos\theta}$$

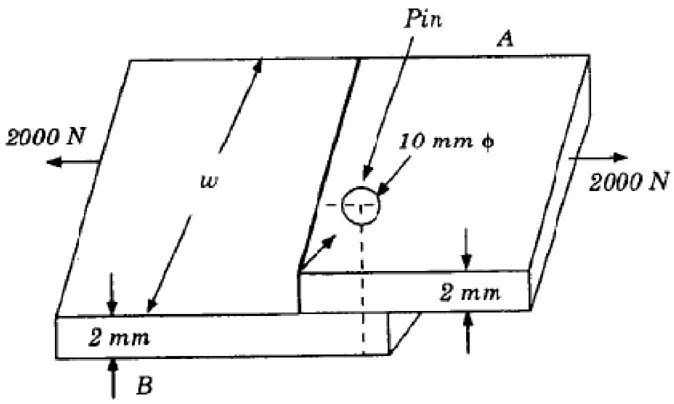
$$= 5\,kN$$



$\therefore$ Shear stress on rivet,

$$\tau = \frac{R}{\text{Area}} = \frac{5 \times 10^3}{500} = 10 \text{ N/mm}^2$$

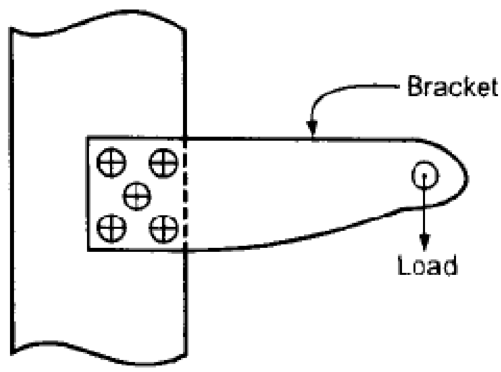
IES-3. **If permissible stress in plates of joint through a pin as shown in the given figure is 200 MPa, then the width w will be**
(a) 15 mm
(b) 18 mm
(c) 20 mm
(d) 25 mm



[IES-1999]

IES-3. Ans. (a) $(w - 10) \times 2 \times 10^{-6} \times 200 \times 10^{6} = 2000$ N; or w = 15 mm.

IES-4. **For the bracket bolted as shown in the figure, the bolts will develop**
(a) Primary tensile stresses and secondary shear stresses
(b) Primary shear stresses and secondary shear stresses
(c) Primary shear stresses and secondary tensile stresses
(d) Primary tensile stresses and secondary compressive stresses



[IES-2000]

IES-4. Ans. (a)

IES-5. **Assertion (A): In pre-loaded bolted joints, there is a tendency for failure to occur in the gross plate section rather than through holes. [IES-2000]**
Reason (R): The effect of pre-loading is to create sufficient friction between the assembled parts so that no slippage occurs.
(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

IES-5. Ans. (a)

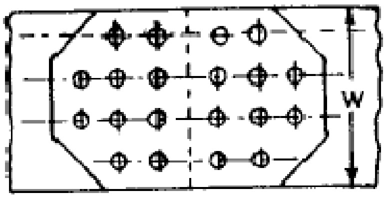
IES-6. Two rigid plates are clamped by means of bolt and nut with an initial force N. After tightening, a separating force P (P < N) is applied to the lower plate, which in turn acts on nut. The tension in the bolt after this is: [IES-1996]

(a) (N + P) (b) (N – P) (c) P (d) N

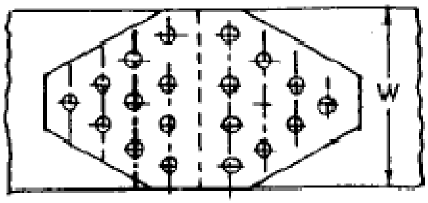
IES-6. Ans. (a)

Efficiency of a riveted joint

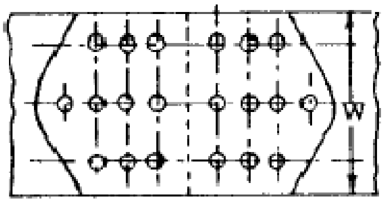
IES-7. Which one of the following structural joints with 10 rivets and same size of plate and material will be the most efficient? [IES-1994]



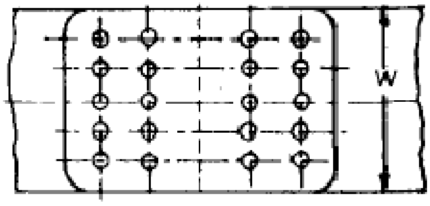
(a)



(b)



(c)



(d)

IES-7. Ans. (b)

IES-8. The most efficient riveted joint possible is one which would be as strong in tension, shear and bearing as the original plates to be joined. But this can never be achieved because: [IES-1993]

(a) Rivets cannot be made with the same material
(b) Rivets are weak in compression
(c) There should be at least one hole in the plate reducing its strength
(d) Clearance is present between the plate and the rivet

IES-8. Ans. (c) Riveted joint can't be as strong as original plates, because there should be at least one hole in the plate reducing its strength.

Advantages and disadvantages of welded joints

IES-9. Assertion (A): In a boiler shell with riveted construction, the longitudinal scam is, jointed by butt joint. [IES-2001]

Reason (R): A butt joint is stronger than a lap joint in a riveted construction.

(a) Both A and R are individually true and R is the correct explanation of A
(b) Both A and R are individually true but R is **NOT** the correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

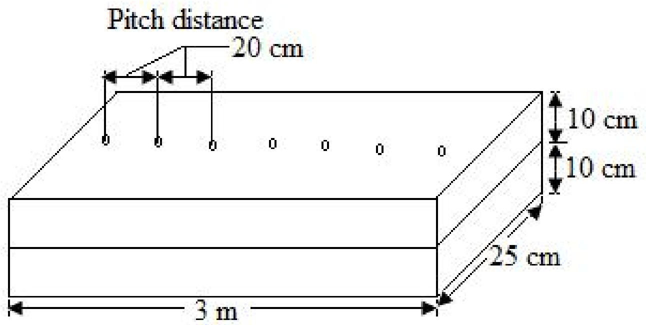
IES-9. Ans. (c)

Previous 20-Years IAS Questions

Failure of riveted joint

IAS-1. **Two identical planks of wood are connected by bolts at a pitch distance of 20 cm. The beam is subjected to a bending moment of 12 kNm, the shear force in the bolts will be:**

(a) Zero (b) 0.1 kN
(c) 0.2 kN (d) 4 kN



[IAS-2001]

IAS-1. Ans. (a)

IAS-2. **Match List-I with List-II and select the correct answer using the code given below the Lists:** **[IAS-2007]**

List-I (Stress Induced)	**List-II (Situation/ Location)**
A. Membrane stress	**1.** Neutral axis of beam
B. Torsional shear stress	**2.** Closed coil helical spring under axial load
C. Double shear stress	**3.** Cylindrical shell subject to fluid pressure
D. Maximum shear stress	**4.** Rivets of double strap butt joint

Code:	**A**	**B**	**C**	**D**		**A**	**B**	**C**	**D**
(a)	3	1	4	2	(b)	4	2	3	1
(c)	3	2	4	1	(d)	4	1	3	2

IAS-2. Ans. (c)

Previous Conventional Questions with Answers

Conventional Question GATE-1994

***Question*:** **The longitudinal joint of a thin cylindrical pressure vessel, 6 m internal diameter and 16 mm plate thickness, is double riveted lap point with no staggering between the rows. The rivets are of 20 mm nominal (diameter with a pitch of 72 mm. What is the efficiency of the joint and what would be the safe pressure inside the vessel? Allowable stresses for the plate and rivet materials are; 145 MN/m² in shear and 230 MN/m² in bearing. Take rivet hole diameter as 1.5 mm more than the rivet diameter.**

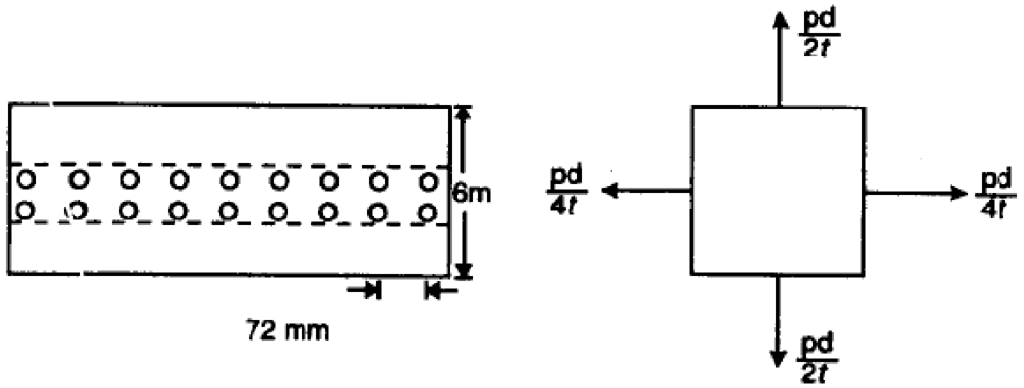
***Answer*:** Given: Diameter of rivet = 20 mm
Diameter of hole = 20 + 1.5 = 21.5 mm
Diameter the pressure vessel, d = 6 m
Thickness of the plate, t = 16 mm
Type of the joint: Double riveted lap joint
Allowable stresses:

$\sigma_1 = 145\,MN/m^2;\ \tau = 120\,MN/m^2;\sigma_c = 230\,MN/m^2$

$$\text{Strength of plate in tearing/pitch}, R_t = \left[\frac{72-(2\times 21.5)}{1000}\right]\times\frac{16}{1000}\times 145$$
$$= 0.06728\ MN$$

$$\text{Strength of rivert in tearing/pitch}, R_s = 2\times\frac{\pi}{4}\times\left(\frac{20}{1000}\right)^2\times 120$$
$$= 0.0754\ MN$$

$$\text{Strength of plate in crushing/pitch}, R_s = 2\times\left(\frac{20}{1000}\times\frac{16}{1000}\right)\times 230$$
$$= 0.1472\ \text{MN}$$



From the above three modes of failure it can be seen that the weakest element is the plate as it will have tear failure at 0.06728 MN/pitch load itself.
Stresses acting on the plate for an inside pressure of pN/m² is shown in figure.

$$\text{Hoop stress} = \frac{pd}{2t} = \frac{p\times 6}{2\times(0.016)} = 187.5p$$

$$\text{Longitudinal stress} = \frac{pd}{4t} = \frac{p\times 6}{4\times(0.016)} = 93.75p$$

$$\text{Maximum principal stress acting on the plate} = \frac{pd}{2t}$$

only$(i,e.187.5\,p)$as there is no shear stress.

$$\text{or}\qquad 187.5\,p \le \frac{0.06728}{(0.016)\times\left[\frac{72-(2\times 21.5)}{1000}\right]} \le 145$$

$$\text{or}\qquad p \le 0.7733\ MN/m^2\ or\ 0.7733\,MPa$$

$$\eta_{joint} = \frac{R_t}{p.t.\sigma_t} = \frac{0.06728}{(0.072)\times(0.016)\times 145} = 0.4028 = 40.28\%$$

Conventional Question GATE-1995

Question: **Determine the shaft diameter and bolt size for a marine flange-coupling transmitting 3.75 MW at 150 r.p.m. The allowable shear stress in the shaft and bolts may be taken as 50 MPa. The number of bolts may be taken as 10 and bolt pitch circle diameter as 1.6 times the shaft diameter.**

Answer: Given, P = 3.75MW; N = 150 r.p.m.;

$\tau_s = \tau_b = 50\,MPa;\qquad n = 10,\qquad D_b = 1.6\,D$

Shaft diameter, D:

$$P = \frac{2\pi NT}{60}$$

E $$3.78\times10^6 = \frac{2\pi\times150\times T}{60}$$

or $$T = \frac{3.75\times10^6\times60}{2\pi\times150} = 238732 \text{ Nm}$$

Also, $$T = \tau_s\times\frac{\pi}{16}\times D^3$$

or $$238732 = 50\times10^6\times\frac{\pi}{16}D^3$$

$\therefore$ $$D = \left(\frac{238732\times16}{50\times10^6\times\pi}\right) = 0.28\,m\ or\ 290 \text{ mm}$$

Bolt size, d_b :

Bolt pitch circle diameter, $D_b = 1.6\,D = 1.6\times0.29 = 0.464\,\text{m}$

Now, $$T = n\times\frac{\pi}{4}d_b^2\times\tau_b\times\left(\frac{D_b}{2}\right)$$

or $$238732 = 10\times\frac{\pi}{4}d_b^2\times50\times10^6\times\left(\frac{0.464}{2}\right)$$

or $$d_b = 0.0512 \text{ m} \ \text{ or } \ 51.2 \text{ mm}$$